

More on Stochastic and Variational Approach to the Lax-Friedrichs Scheme

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Abstract

The author presented a stochastic and variational aspect of the Lax-Friedrichs scheme applied to hyperbolic scalar conservation laws [18]. We will extend the results of [18] on the Lax-Friedrichs scheme, showing its time global stability; its long time behaviors; its error estimates. The proofs essentially rely on calculus of variations in the Lax-Friedrichs scheme and viscosity solutions of Hamilton-Jacobi equations corresponding to the hyperbolic scalar conservation laws. We also provide basic facts that are useful in numerical analysis and simulation of the weak KAM theory. As an application, we show rigorous treatment of finite difference approximation to KAM tori.

Keywords: Lax-Friedrichs scheme; scalar conservation law; Hamilton-Jacobi equation; calculus of variations; random walk; weak KAM theory

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1 Introduction

We investigate the Lax-Friedrichs scheme applied to initial value problems of hyperbolic scalar conservation laws with a constant c

$$(1.1) \quad u_t + H(x, t, c + u)_x = 0.$$

There is the huge literature on stability and convergence of the scheme. The standard technique is based on the L^1 -framework with a priori estimates and the compactness of functions of bounded variation, where mesh size independent boundedness of both difference solutions and their total variation must be verified. Since the Lax-Friedrichs scheme is very simple, one can successfully analyze details of approximation, particularly in the case where a flux function is of the simple form $H(x, t, p) = H(p)$. We refer to [5], [19], [15] and the works cited there. However, in the case of a general flux function depending on both x and t , the problem becomes much harder and often requires unpleasant assumptions. The results of the general case first appear in [14], where stability and L^1 -convergence are proved with a restricted time interval that is determined by the growth of $H(x, t, p)$ with respect to p . In [13], time-global stability

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and L^1 -convergence within arbitrary time intervals are proved for a flux function of the form $H(x, t, p) = f(p) + F(x, t)$ under the periodic setting, with many details of large time behaviors of the Lax-Friedrichs scheme. Still it seems very hard to obtain results similar to [13] for more general flux functions by the standard approach based on the L^1 -framework.

Recently the author announced a stochastic and variational approach to the Lax-Friedrichs scheme [18], where stability and convergence are proved based on the *low of large numbers* in hyperbolic scaling limit of random walks and the *calculus of variations* in the theory of viscosity solutions of the Hamilton-Jacobi equations with constants c and $h(c)$

$$(1.2) \quad v_t + H(x, t, c + v_x) = h(c).$$

This is a finite difference version of the work [9] on the stochastic and variational approach to the vanishing viscosity method. Now we briefly refer to our stochastic and variational approach. Consider initial value problems of the inviscid hyperbolic scalar conservation law and the corresponding Hamilton-Jacobi equation

$$(1.3) \quad \begin{cases} u_t + H(x, t, c + u_x) = 0 & \text{in } \mathbb{T} \times (0, T], \\ u(x, 0) = u^0(x) \in L^\infty(\mathbb{T}) & \text{on } \mathbb{T}, \quad \int_{\mathbb{T}} u^0(x) dx = 0, \quad \|u^0\|_{L^\infty} \leq r, \end{cases}$$

$$(1.4) \quad \begin{cases} v_t + H(x, t, c + v_x) = h(c) & \text{in } \mathbb{T} \times (0, T], \\ v(x, 0) = v^0(x) \in Lip(\mathbb{T}) & \text{on } \mathbb{T}, \quad \|v_x^0\|_{L^\infty} \leq r, \end{cases}$$

where c is a parameter varying within an interval $[c_0, c_1]$, $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the standard torus, $h(c)$ is a continuous function and $r > 0$ is a constant. We arbitrarily fix T , r and $[c_0, c_1]$. Note that (1.3) and (1.4) is equivalent in the sense that the entropy solution u or viscosity solution v is derived from the other if $u^0 = v_x^0$. In particular we have $u = v_x$ (see e.g. [1]). We always assume that $u^0 = v_x^0$. Our flux function H is assumed to satisfy the following (A1)-(A4):

$$(A1) \ H(x, t, p) : \mathbb{T}^2 \times \mathbb{R} \rightarrow \mathbb{R}, \ C^2 \quad (A2) \ H_{pp} > 0 \quad (A3) \ \lim_{|p| \rightarrow +\infty} \frac{H(x, t, p)}{|p|} = +\infty.$$

By (A1)-(A3), we have the Legendre transform $L(x, t, \xi)$ of $H(x, t, \cdot)$, which is now given by

$$L(x, t, \xi) = \sup_{p \in \mathbb{R}} \{\xi p - H(x, t, p)\}$$

and satisfies

$$(A1)' \ L(x, t, \xi) : \mathbb{T}^2 \times \mathbb{R} \rightarrow \mathbb{R}, \ C^2 \quad (A2)' \ L_{\xi\xi} > 0 \quad (A3)' \ \lim_{|\xi| \rightarrow +\infty} \frac{L(x, t, \xi)}{|\xi|} = +\infty.$$

The last assumption is

$$(A4) \ \text{There exists } \alpha > 0 \text{ such that } |L_x| \leq \alpha(|L| + 1).$$

We discretize the equation of (1.3) by the Lax-Friedrichs scheme as follows:

$$(1.5) \quad \frac{u_{m+1}^{k+1} - \frac{(u_m^k + u_{m+2}^k)}{2}}{\Delta t} + \frac{H(x_{m+2}, t_k, c + u_{m+2}^k) - H(x_m, t_k, c + u_m^k)}{2\Delta x} = 0.$$

We also discretize the equation of (1.4) by the following scheme

$$(1.6) \quad \frac{v_m^{k+1} - \frac{(v_{m-1}^k + v_{m+1}^k)}{2}}{\Delta t} + H(x_m, t_k, c + \frac{v_{m+1}^k - v_{m-1}^k}{2\Delta x}) = h(c).$$

Note that (1.5) and (1.6) are also equivalent in the sense that u_m^k or v_{m+1}^k is derived from the other. In particular we have

$$u_m^k = \frac{v_{m+1}^k - v_{m-1}^k}{2\Delta x},$$

which is an important relation in this paper. In the stochastic and variational approach, “stochastic” comes from the numerical viscosity intrinsic to (1.5) and (1.6), and “variational” from the variational structures of Hamilton-Jacobi equations. We state several points of the stochastic and variational approach [18]:

- (1) Stochastic and variational representation formulas (value functions) for v_{m+1}^k and u_m^k are given.
- (2) Stability of the Lax-Friedrichs scheme up to arbitrary $T > 0$ is derived from variational techniques.
- (3) Almost everywhere pointwise convergence of u_m^k to $u = v_x$ is proved. In particular this yields the uniform convergence except neighborhoods of shocks with arbitrarily small measure.
- (4) Uniform convergence of v_{m+1}^k to v with an error $O(\sqrt{\Delta x})$ is proved from a stochastic viewpoint.
- (5) Random walks play a role as “characteristic curves” of the difference equations, which converge to the genuine ones.

The purpose of this paper is to show further results on the Lax-Friedrichs scheme, under the assumptions (A1)-(A4), based on (1)-(5) and with techniques of the theory of viscosity solutions of Hamilton-Jacobi equations. Although our arguments here are on the Lax-Friedrichs scheme, other finite difference schemes with the numerical viscosity are available as well. The main results are on

1. Time global stability of the Lax-Friedrichs scheme with a fixed meshed size.
2. Error estimates for entropy solutions.

It is proved that genuine entropy solutions at $t = 1$ are uniformly bounded, which is independent of the magnitude of initial datas. Since difference entropy solutions are “close” to the genuine ones for small mesh size, difference entropy solutions at $t = 1$ are also uniformly bounded. Due to the periodic setting, iteration of time-1 analysis yields time global properties. Combining these facts, we obtain time global stability of the Lax-Friedrichs scheme. As a result, we can show the long time behavior of the Lax-Friedrichs scheme that any solutions associated with c fall into the time periodic state uniquely determined by each c , which means that we obtain the unique space-time periodic difference entropy solution and the unique, up to constant, space-time periodic difference viscosity solution for each c . They approximate a genuine \mathbb{Z}^2 -periodic

entropy/viscosity solution of (1.1)/(1.2). In the periodic states, we naturally have the notion of the effective Hamiltonian for difference Hamilton-Jacobi equation (1.6). We also prove convergence with an error estimates of the effective Hamiltonian to the genuine one, showing its properties.

It is known that the optimal L^1 -error estimate between u_m^k and u is $O(\sqrt{\Delta x})$ in the case of $H(x, t, p) = H(p)$ [15]. The upper bound $O(\sqrt{\Delta x})$ is due to properties of functions of bounded variation [11]. It is not clear whether [11] is applicable to the case of our general flux functions. We show, with different approach, an L^1 -error estimate of $O(\Delta x^{\frac{1}{4}})$. Our error estimate is based on $O(\sqrt{\Delta x})$ which arises as an “error” between random walks for the Lax-Friedrichs scheme and their space-time continuous limit under hyperbolic scaling (that is a backward characteristic curve). For a technical reason, we further lose $1/4$ in the case of general flux function $H(x, t, p)$. In addition we show that, if the genuine entropy solution is Lipschitz, a C^0 -error estimate of $O(\Delta x^{\frac{1}{4}})$ is available.

Unlike the case of initial value problems, it is challenging to show convergence of full sequences and an error estimate for \mathbb{Z}^2 -periodic entropy/viscosity solutions of (1.1)/(1.2), because uniqueness of genuine \mathbb{Z}^2 -periodic solutions with respect to c is not valid in general. However we can manage the special case where a genuine \mathbb{Z}^2 -periodic entropy solution \bar{u} with some c is C^1 and the dynamics of its characteristic curves $C^*(s) := (q(s) \bmod 1, s \bmod 1)$ are C^1 -conjugate to that of the linear flow on \mathbb{T}^2 with a Diophantine rotation vector. Existence of such a solution \bar{u} is known as a *KAM* torus in Hamiltonian dynamics (see e.g. [12], [16], [10]). We show a C^0 -error estimate depending on the Diophantine condition of the rotation vector, which is a result on approximation of KAM tori. Our proof is based on the fact that one orbit of the linear flow on \mathbb{T}^2 with a Diophantine rotation vector is ergodic on \mathbb{T}^2 . It may be impossible to obtain the same result without our stochastic and variational approach to the Lax-Friedrichs scheme, because if so there is no way to track the genuine characteristic curves, namely the result relies on the fact that random walks are close to the genuine ergodic characteristic curves.

Finally we state that our motivation comes from not only continuum mechanics but also a recent theory of Lagrangian and Hamiltonian dynamics called the Aubry-Mather theory or the weak KAM theory (also from the classical KAM theory) [8], [10], [7]. Our periodic setting is standard and \mathbb{Z}^2 -periodic entropy/viscosity solutions of (1.1)/(1.2) and the effective Hamiltonian play central roles in the weak KAM theory. The results of this paper provide basic tools of numerical analysis of the weak KAM theory through finite difference approximation. We remark that it is better to approximate entropy solutions and characteristic curves as well as viscosity solutions from the view point of accuracy, because the central objects in the weak KAM theory such as KAM tori, Aubry-Mather sets, effective Hamiltonians, calibrated curves etc. are derived from the derivative of viscosity solutions or entropy solutions. The “derivative” of numerical viscosity solutions obtained through a scheme which has no relation to entropy solutions is not accurate in general. Some efforts in finite difference approximation to the weak KAM theory are found in [13]. However due to the absence of the stochastic and variational approach to the Lax-Friedrichs scheme, the results are mathematically restricted. We also point out [10], [2] for results on smooth approximation methods of the weak KAM theory by the vanishing viscosity method. In particular [2] successfully exploits the stochastic and variational approach to the vanishing viscosity method given in [9], where stochastic

ODEs with the standard Brownian motion approximate the genuine characteristic curves. Our results allow ones to develop more on finite difference approximation methods of the weak KAM theory.

2 Preliminary Result

We state several important background or preliminary results.

2.1 Entropy Solution and Viscosity Solution

It is well-known that the viscosity solution v of (1.4), which is Lipschitz, is characterized by calculus of variations: The value of v at each point $(x, t) \in \mathbb{T} \times (0, T]$, $T \in (0, \infty)$ is given by

$$(2.1) \quad v(x, t) = \inf_{\gamma \in AC, \gamma(t)=x} \left\{ \int_0^t L^c(\gamma(s), s, \gamma'(s)) ds + v^0(\gamma(0)) \right\} + h(c)t,$$

where AC is the family of absolutely continuous curves $\gamma : [0, t] \rightarrow \mathbb{T}$ and

$$L^c(x, t, \xi) := L(x, t, \xi) - c\xi$$

is the Legendre transform of $H(x, t, c + \cdot)$. We can find a minimizing curve γ^* of (2.1), which is a backward characteristic curve of (1.1) and (1.2), and a C^2 -solution of the Euler-Lagrange equation generated by the Lagrangian L^c . On each minimizing curve, v is differentiable with respect to x :

$$(2.2) \quad v_x(\gamma^*(s), s) = L_\xi^c(\gamma^*(s), s, \gamma^{*'}(s)) \text{ for } 0 < s < t.$$

We say that a point (x, t) is a regular point of v or just regular, if there exists $v_x(x, t)$. Since v is Lipschitz, almost every points are regular. In particular, if (x, t) is regular, a minimizing curve γ^* for (2.1) is unique and (2.2) holds for $s = t$.

Usually the entropy solution u of (1.3) is defined as an element of $C^0((0, T]; L^1(\mathbb{T}))$. Here we always take the representative element of such u given by v_x , which is still denoted by u . If (x, t) is regular and γ^* is the unique minimizing curve for $v(x, t)$, the value of the entropy solution $u = v_x$ at the point (x, t) is given by

$$u(x, t) = \int_0^t L_x^c(\gamma^*(s), s, \gamma^{*'}(s)) ds + u^0(\gamma^*(0)),$$

where u^0 is supposed to be rarefaction-free, or equivalently v^0 is semiconcave. Otherwise $u^0(\gamma^*(0))$ needs to be replaced with $L_\xi^c(\gamma^*(0), 0, \gamma^{*'}(0))$. In particular we have for any $\tau \in [0, t)$

$$u(x, t) = \int_\tau^t L_x^c(\gamma^*(s), s, \gamma^{*'}(s)) ds + L_\xi^c(\gamma^*(\tau), \tau, \gamma^{*'}(\tau)).$$

For more details, see e.g. [1], [4].

We introduce the solution operators of (1.3) and (1.4)

$$\phi^t : L_{r,0}^\infty(\mathbb{T}) \ni u^0 \mapsto u(\cdot, t) \in L^\infty(\mathbb{T}), \quad \psi^t : Lip_r(\mathbb{T}) \ni v^0 \mapsto v(\cdot, t) \in Lip(\mathbb{T}),$$

where $L_{r,0}^\infty(\mathbb{T})$ is the set of all functions $u^0 \in L^\infty(\mathbb{T})$ with $\|u^0\|_{L^\infty} \leq r$ and $\int_{\mathbb{T}} u^0 dx = 0$, and $Lip_r(\mathbb{T})$ is the set of all Lipschitz functions on \mathbb{T} with a Lipschitz constant bounded by r . When we specify the value of c , we write $\phi^t(\cdot; c)$, $\psi_\Delta^t(\cdot; c)$, u^c, v^c .

We would like to prove an a priori boundedness of $u(x, t) = v_x(x, t)$.

Proposition 2.1. *For each $t \in (0, T]$, there exists a constant $\beta_1(t) > 0$ independent of $r, c \in [c_0, c_1]$ and initial datas v^0, u^0 for which we have*

$$\|\phi^t(u^0; c)\|_{L^\infty} \leq \beta_1(t), \quad \|\psi^t(v^0; c)_x\|_{L^\infty} \leq \beta_1(t).$$

Proof. Fix $t \in (0, T]$. Due to (2.2), which holds for $s = t$ if (x, t) is regular, it is enough to estimate $L_\xi^c(\gamma^*(t), t, \gamma^{*'}(t))$ for each minimizing curve γ^* . We prepare two lemmas.

Lemma 2.2. *Let γ^* be a minimizing curve for $v(x, t)$. Set $y := \gamma^*(0)$. Then γ^* attains*

$$\inf_{\gamma \in AC, \gamma(t)=x, \gamma(0)=y} \int_0^t L^c(\gamma(s), s, \gamma'(s)) ds.$$

Proof. If not, there exists γ^\sharp such that

$$\int_0^t L^c(\gamma^\sharp(s), s, \gamma^{\sharp'}(s)) ds < \int_0^t L^c(\gamma^*(s), s, \gamma^{*'}(s)) ds.$$

Since $v^0(\gamma^\sharp(0)) = y = v^0(\gamma^*(0))$, we have

$$\int_0^t L^c(\gamma^\sharp(s), s, \gamma^{\sharp'}(s)) ds + v^0(\gamma^\sharp(0)) < \int_0^t L^c(\gamma^*(s), s, \gamma^{*'}(s)) ds + v^0(\gamma^*(0)).$$

Therefore γ^* is not a minimizing curve for $v(x, t)$, which is a contradiction. \square

Define the set

$$\Gamma(t) := \left\{ \gamma^c \mid \gamma^c \text{ attains } \inf_{\gamma(t)=x, \gamma(0)=y} \int_0^t L^c(\gamma(s), s, \gamma'(s)) ds, \ x, y \in \mathbb{T}, c \in [c_0, c_1] \right\}.$$

By Lemma 2.2, any minimizing curve γ^* for $v(x, t)$, $x \in \mathbb{T}$ belongs to $\Gamma(t)$. (To be precise, we should take $\gamma^* \bmod 1$, but this is not important due to the periodic setting.)

Lemma 2.3. *1. There exists a constant $C_1(t) > 0$ such that for any $x, y \in \mathbb{T}$ we have a C^1 -curve γ which satisfies*

$$\gamma(t) = x, \quad \gamma(0) = y, \quad \int_0^t L^c(\gamma(s), s, \gamma'(s)) ds \leq C_1(t).$$

In Particular, any $\gamma^c \in \Gamma(t)$ satisfies

$$\int_0^t L^c(\gamma^c(s), s, \gamma^{c'}(s)) ds \leq C_1(t)$$

2. There exists a constant $C_2(t) > 0$ such that for any $\gamma^c \in \Gamma(t)$ we have $\tau \in (0, t)$ which satisfies

$$|\gamma^{c'}(\tau)| \leq C_2(t).$$

3. There exists a constant $C_3(t) > 0$ such that for any $\gamma^c \in \Gamma(t)$ we have

$$|L_\xi^c(\gamma^c(s), s, \gamma^{c'}(s))| \leq C_3(t), \quad s \in [0, t].$$

Proof. 1. Consider $\gamma(s) := x + \frac{x-y}{t}(s-t)$. Since $|x-y| \leq 1$, we have $|\gamma'(s)| \leq t^{-1}$. Therefore we obtain

$$\int_0^t L^c(\gamma(s), s, \gamma'(s)) ds \leq \sup_{x, s \in \mathbb{T}, |\xi| \leq t^{-1}, c \in [c_0, c_1]} |L^c(x, s, \xi)| t.$$

Set $C_1(t) := \sup_{x, s \in \mathbb{T}, |\xi| \leq t^{-1}, c \in [c_0, c_1]} |L^c(x, s, \xi)| t$ and 1. is proved.

2. We have $\tau \in (0, t)$ which satisfies, due to 1. and the minimizing property of γ^c ,

$$C_1(t) \geq \int_0^t L^c(\gamma^c(s), s, \gamma^{c'}(s)) ds = L^c(\gamma^c(\tau), \tau, \gamma^{c'}(\tau)) t.$$

By (A3), $|\gamma^{c'}(\tau)|$ must be bounded by a constant $C_2(t)$ independent of $\gamma^c \in \Gamma(t)$.

3. Note that γ^c is a C^2 -solution of the Euler-Lagrange equation generated by L^c

$$\frac{d}{dt} L_\xi^c(\gamma^c(s), s, \gamma^{c'}(s)) = L_x(\gamma^c(s), s, \gamma^{c'}(s)).$$

It follows from (A1)-(A4) that there exists α_1 for which we have $|L_x^c| \leq \alpha_1(|L^c| + 1)$ for any $c \in [c_0, c_1]$, and that $L_* := |\min\{0, \inf_{x, s, \xi, c} L^c\}|$ is bounded. We have $\tau^* \in [0, t]$ which attains the maximum of $|L_\xi^c(\gamma^c(s), s, \gamma^{c'}(s))|$. Suppose that $\tau^* \neq \tau$, where τ is the number in 2.

$$\begin{aligned} \left| \int_\tau^{\tau^*} \frac{d}{dt} L_\xi^c(\gamma^c(s), s, \gamma^{c'}(s)) ds \right| &= |L_\xi^c(\gamma^c(\tau^*), \tau^*, \gamma^{c'}(\tau^*)) - L_\xi^c(\gamma^c(\tau), \tau, \gamma^{c'}(\tau))| \\ &\leq \int_0^t |L_x^c(\gamma^c(s), s, \gamma^{c'}(s))| ds \\ &\leq \int_0^t \alpha_1 (1 + |L^c(\gamma^c(s), s, \gamma^{c'}(s))|) ds \\ &\leq \alpha_1 \int_0^t 1 + (L^c(\gamma^c(s), s, \gamma^{c'}(s)) + L_*) + L_* ds \\ &= \alpha_1 (2L_* + 1)t + \alpha_1 \int_0^t L^c(\gamma^c(s), s, \gamma^{c'}(s)) ds \\ &\leq \alpha_1 (2L_* + 1)t + \alpha_1 C_1(t). \end{aligned}$$

Therefore, setting

$$C_3(t) := \alpha_1 (2L_* + 1)t + \alpha_1 C_1(t) + \sup_{x, s \in \mathbb{T}, |\xi| \leq C_2(t), c \in [c_0, c_1]} |L_\xi^c(x, s, \xi)|,$$

we obtain for $0 \leq s \leq t$

$$|L_\xi^c(\gamma^c(s), s, \gamma^{c'}(s))| \leq |L_\xi^c(\gamma^c(\tau^*), \tau^*, \gamma^{c'}(\tau^*))| \leq C_3(t).$$

The case where $\tau^* = \tau$ is included by the above inequality. □

Since (x, t) is regular for almost every $x \in \mathbb{T}$ with each fixed t and $v_x(x, t) = L_\xi^c(\gamma^*(t), t, \gamma^{*'}(t))$ holds almost every $x \in \mathbb{T}$, we obtain Proposition 2.1 by setting $\beta_1(t) := C_3(t)$. \square

We show continuity of $\phi^t(v_x^0; c)$ and $\psi^t(v_0; c)$ with respect to v^0 and c .

Proposition 2.4. *Fix $t \in (0, T]$. For each sequence $v_j^0 \rightarrow v^0$ uniformly and $c^j \rightarrow c$ as $j \rightarrow \infty$ ($v_{j,x}^0$ is not necessarily convergent), we have*

$$\psi^t(v_j^0; c^j) \rightarrow \psi^t(v^0; c) \text{ uniformly, } \phi^t(v_{j,x}^0; c^j) \rightarrow \phi^t(v_x^0; c) \text{ in } L^1(\mathbb{T}) \quad \text{as } j \rightarrow \infty.$$

Proof. By the variational representation, we have

$$\begin{aligned} \psi^t(v^0; c)(x) &= \int_0^t L(\gamma^*(s), s, \gamma^{*'}(s)) - c\gamma^{*'}(s) ds + v^0(\gamma^*(0)) + h(c)t, \\ \psi^t(v_j^0; c^j)(x) &= \int_0^t L(\gamma_j^*(s), s, \gamma_j^{*'}(s)) - c^j\gamma_j^{*'}(s) ds + v_j^0(\gamma_j^*(0)) + h(c^j)t \end{aligned}$$

and hence

$$\begin{aligned} \psi^t(v_j^0; c^j)(x) - \psi^t(v^0; c)(x) &\leq \int_0^t -(c^j - c)\gamma^{*'}(s) ds + v_j^0(\gamma^*(0)) - v^0(\gamma^*(0)) \\ &\quad + (h(c^j) - h(c))t, \\ \psi^t(v_j^0; c^j)(x) - \psi^t(v^0; c)(x) &\geq \int_0^t -(c^j - c)\gamma_j^{*'}(s) ds + v_j^0(\gamma_j^*(0)) - v^0(\gamma_j^*(0)) \\ &\quad + (h(c^j) - h(c))t. \end{aligned}$$

It follows from 3. of Lemma, 2.3 that any minimizing curves for $v(x, t)$ are Lipschitz with a common Lipschitz constant for all $x \in \mathbb{T}$ and $v^0 \in Lip_r(\mathbb{T})$. Since h is continuous, we conclude $\psi^t(v_j^0; c^j) \rightarrow \psi^t(v^0; c)$ uniformly as $j \rightarrow \infty$.

Let $x \in \mathbb{T}$ be a common regular point of all $\psi^t(v_j^0; c^j)$, $j = 1, 2, 3, \dots$. Almost every point is such. Due to a variational technique, we see that $\gamma_j^* \rightarrow \gamma^*$ uniformly and $\gamma_j^{*'} \rightarrow \gamma^{*'}$ in L^2 as $j \rightarrow \infty$ (see e.g. [18], Lemma 3.4). Note that for each $0 \leq \tau < t$

$$\begin{aligned} \phi^t(v_x^0; c)(x) &= \psi^t(v^0; c)_x(x) = \int_\tau^t L_x(\gamma^*(s), s, \gamma^{*'}(s)) ds + L_\xi(\gamma^*(\tau), \tau, \gamma^{*'}(\tau)) - c, \\ \phi^t(v_{j,x}^0; c^j)(x) &= \psi^t(v_j^0; c^j)_x(x) = \int_\tau^t L_x(\gamma_j^*(s), s, \gamma_j^{*'}(s)) ds + L_\xi(\gamma_j^*(\tau), \tau, \gamma_j^{*'}(\tau)) - c^j. \end{aligned}$$

For any $\varepsilon > 0$, there exists J such that, if $j \geq J$, we have $\|\gamma_j^* - \gamma^*\|_{C^0} \leq \varepsilon$, $\|\gamma_j^{*'} - \gamma^{*'}\|_{L^2} \leq \varepsilon\sqrt{t}$. Note that we have τ depending on $j \geq J$ such that $|\gamma_j^{*'}(\tau) - \gamma^{*'}(\tau)| \leq \varepsilon$. Therefore we conclude $\phi^t(v_{j,x}^0; c^j) \rightarrow \phi^t(v_x^0; c)$ pointwise almost everywhere. This immediately leads to $L^1(\mathbb{T})$ -convergence. \square

2.2 Stochastic and Variational Approach to the Lax-Friedrichs scheme

We state several results of the stochastic and variational approach to the Lax-Friedrichs scheme shown in [18]. Let N, K be natural numbers with $N \leq K$. The mesh size

$\Delta = (\Delta x, \Delta t)$ is defined by $\Delta x := (2N)^{-1}$ and $\Delta t := (2K)^{-1}$. Set $\lambda := \Delta t / \Delta x$, $x_m := m\Delta x$ for $m \in \mathbb{Z}$ and $t_k := k\Delta t$ for $k = 0, 1, 2, \dots$. For $x \in \mathbb{R}$ and $t > 0$, the notation $m(x), k(t)$ denote the integers m, k for which $x \in [x_m, x_m + 2\Delta x), t \in [t_k, t_k + \Delta t)$. Let $(\Delta x \mathbb{Z}) \times (\Delta t \mathbb{Z}_{\geq 0})$ be the set of all (x_m, t_k) and

$$\mathcal{G}_{\text{even}} \subset (\Delta x \mathbb{Z}) \times (\Delta t \mathbb{Z}_{\geq 0}), \quad \mathcal{G}_{\text{odd}} \subset (\Delta x \mathbb{Z}) \times (\Delta t \mathbb{Z}_{\geq 0})$$

be the set of all (x_m, t_k) with $k = 0, 1, 2, \dots$ and $m \in \mathbb{Z}$ such that $m + k = \text{even, odd}$. We call $\mathcal{G}_{\text{even}}, \mathcal{G}_{\text{odd}}$ the even-grid, odd-grid. We consider the discretization of (1.3) by the Lax-Freidrichs scheme in $\mathcal{G}_{\text{even}}$:

$$(2.3) \quad \begin{cases} \frac{u_{m+1}^{k+1} - \frac{(u_m^k + u_{m+2}^k)}{2}}{\Delta t} + \frac{H(x_{m+2}, t_k, c + u_{m+2}^k) - H(x_m, t_k, c + u_m^k)}{2\Delta x} = 0, \\ u_m^0 = u_{\Delta}^0(x_m), \quad u_{m \pm 2N}^k = u_m^k, \end{cases}$$

where for $m = \text{even}$

$$(2.4) \quad u_{\Delta}^0(x) := \frac{1}{2\Delta x} \int_{x_m - \Delta x}^{x_m + \Delta x} u^0(y) dy \quad \text{for } x \in [x_m - \Delta x, x_m + \Delta x).$$

Note that $\sum_{\{m \mid 0 \leq m < 2N, m+k=\text{even}\}} u_m^k \cdot 2\Delta x$ is conservative with respect to k and is zero for u^0 which has the zero-average. We also discretize (1.4) in \mathcal{G}_{odd} :

$$(2.5) \quad \begin{cases} \frac{v_m^{k+1} - \frac{(v_{m-1}^k + v_{m+1}^k)}{2}}{\Delta t} + H(x_m, t_k, c + \frac{v_{m+1}^k - v_{m-1}^k}{2\Delta x}) = h(c), \\ v_{m+1}^0 = v_{\Delta}^0(x_{m+1}), \quad v_{m+1 \pm 2N}^k = v_{m+1}^k, \end{cases}$$

where, in addition to the assumption $u^0 = v_x^0$, we assume that

$$(2.6) \quad v_{\Delta}^0(x) := v^0(-\Delta x) + \int_{-\Delta x}^x u_{\Delta}^0(y) dy \quad (v_{\Delta}^0(x_{m+1}) = v^0(x_{m+1}) \text{ for } m = \text{even}).$$

Note that $u_{\Delta}^0 \rightarrow u^0$ in $L^1(\mathbb{T})$ and $v_{\Delta}^0 \rightarrow v^0$ uniformly with $\|v_{\Delta}^0 - v^0\|_{C^0} \leq \|u^0\|_{L^{\infty}} \cdot 2\Delta x$, as $\Delta \rightarrow 0$. We introduce the difference operators:

$$D_t w_m^{k+1} := \frac{w_m^{k+1} - \frac{(w_{m-1}^k + w_{m+1}^k)}{2}}{\Delta t}, \quad D_x w_{m+1}^k := \frac{w_{m+1}^k - w_{m-1}^k}{2\Delta x}.$$

The two problems (2.3) and (2.5) are equivalent under (2.4) and (2.6). In particular we have $D_x v_{m+1}^k = u_m^k$ [18]. Let u_{Δ} be the step function derived from the solution u_m^k of (2.3), namely

$$u_{\Delta}(x, t) := u_m^k \text{ for } (x, t) \in [x_{m-1}, x_{m+1}) \times [t_k, t_{k+1}).$$

Let v_{Δ} be the linear interpolation with respect to the space variable derived from the solution v_{m+1}^k of (2.5), namely

$$v_{\Delta}(x, t) := v_{m-1}^k + D_x v_{m+1}^k \cdot (x - x_{m-1}) \text{ for } (x, t) \in [x_{m-1}, x_{m+1}) \times [t_k, t_{k+1}).$$

We remark that $v_\Delta(x, \cdot)$ is a step function for each fixed x and that $(v_\Delta)_x = u_\Delta$.

We introduce space-time inhomogeneous random walks in \mathcal{G}_{odd} , which correspond to characteristic curves of (1.3) and (1.4). For each point $(x_n, t_{l+1}) \in \mathcal{G}_{odd}$, we introduce backward random walks γ which start from x_n at t_{l+1} and move by $\pm\Delta x$ in each backward time step:

$$\gamma = \{\gamma^k\}_{k=0,1,\dots,l+1}, \quad \gamma^{l+1} = x_n, \quad \gamma^{k+1} - \gamma^k = \pm\Delta x.$$

More precisely, we introduce the following for each $(x_n, t_{l+1}) \in \mathcal{G}_{odd}$:

$$\begin{aligned} X^k &:= \{x_m \mid (x_m, t_k) \in \mathcal{G}_{odd}, |x_m - x_n| \leq (l+1-k)\Delta x\} \text{ for } k \leq l+1, \\ G &:= \bigcup_{1 \leq k \leq l+1} (X^k \times \{t_k\}) \subset \mathcal{G}_{odd}, \\ \xi &: G \ni (x_m, t_k) \mapsto \xi_m^k \in [-\lambda^{-1}, \lambda^{-1}], \quad \lambda = \Delta t / \Delta x, \\ \bar{\rho} &: G \ni (x_m, t_k) \mapsto \bar{\rho}_m^k := \frac{1}{2} - \frac{1}{2}\lambda \xi_m^k \in [0, 1], \\ \bar{\rho} &: G \ni (x_m, t_k) \mapsto \bar{\rho}_m^k := \frac{1}{2} + \frac{1}{2}\lambda \xi_m^k \in [0, 1], \\ \gamma &: \{0, 1, 2, \dots, l+1\} \ni k \mapsto \gamma^k \in X^k, \quad \gamma^{l+1} = x_n, \quad \gamma^{k+1} - \gamma^k = \pm\Delta x, \\ \Omega &: \text{family of the above } \gamma. \end{aligned}$$

We regard $\bar{\rho}_m^k$ (respectively $\bar{\rho}_m^k$) as a transition probability from (x_m, t_k) to $(x_m + \Delta x, t_k - \Delta t)$ (from (x_m, t_k) to $(x_m - \Delta x, t_k - \Delta t)$). Note that ξ is a control of random walks, which plays a role of a velocity field on the grid. We define the density of each path $\gamma \in \Omega$ as

$$\mu(\gamma) := \prod_{1 \leq k \leq l+1} \rho(\gamma^k, \gamma^{k-1}),$$

where $\rho(\gamma^k, \gamma^{k-1}) = \bar{\rho}_{m(\gamma^k)}^k$ (respectively $\bar{\rho}_{m(\gamma^k)}^k$) if $\gamma^k - \gamma^{k-1} = -\Delta x$ (Δx). The density $\mu(\cdot) = \mu(\cdot; \xi)$ yields a probability measure of Ω , namely

$$prob(A) = \sum_{\gamma \in A} \mu(\gamma; \xi) \quad \text{for } A \subset \Omega.$$

The expectation with respect to this probability measure is denoted by $E_{\mu(\cdot; \xi)}$, namely for a random variable $f : \Omega \rightarrow \mathbb{R}$

$$E_{\mu(\cdot; \xi)}[f(\gamma)] := \sum_{\gamma \in \Omega} \mu(\gamma; \xi) f(\gamma).$$

We use the notation γ as the symbol of random walks or a sample path. If necessary, we write $\gamma = \gamma(x_n, t_{l+1}; \xi)$ in order to specify its initial point and control.

We state an important result on scaling limit of inhomogeneous random walks. Let $\eta(\gamma) = \{\eta^k(\gamma)\}_{k=0,1,2,\dots,l+1}$, $\gamma \in \Omega$ be a random variable which is induced by a random walk $\gamma = \gamma(x_n, t_{l+1}; \xi)$ and is defined as

$$\eta^{l+1} := \gamma^{l+1}, \quad \eta^k(\gamma) := \gamma^{l+1} - \sum_{k < k' \leq l+1} \xi(\gamma^{k'}, t_{k'}) \Delta t \quad \text{for } 0 \leq k \leq l.$$

Proposition 2.5. ([17]) Set $\tilde{\sigma}^k := E_{\mu(\cdot; \xi)}[|\gamma^k - \eta^k(\gamma)|^2]$ and $\tilde{d}^k := E_{\mu(\cdot; \xi)}[|\gamma^k - \eta^k(\gamma)|]$ for $0 \leq k \leq l+1$. Then we have

$$(\tilde{d}^k)^2 \leq \tilde{\sigma}^k \leq \frac{t^{l+1} - t^k}{\lambda} \Delta x.$$

If we take hyperbolic scaling limit, namely $\Delta = (\Delta x, \Delta t) \rightarrow 0$ under

$$0 < \lambda_0 \leq \lambda = \Delta t / \Delta x < \lambda_1,$$

\tilde{d}^k and $\sqrt{\tilde{\sigma}^k}$ always tend to 0 with the order $O(\sqrt{\Delta x})$. Note that the variance does not necessarily do so for inhomogeneous random walks. We refer to [17] for more details of hyperbolic scaling limit of inhomogeneous random walks. We always take the limit $\Delta \rightarrow 0$ under hyperbolic scaling.

Now we state results on the stochastic and variational approach to the Lax-Friedrichs scheme:

Theorem 2.6 ([18]). *There exists $\lambda_1 > 0$ (depending on T , $[c_0, c_1]$ and r) such that for any small $\Delta = (\Delta x, \Delta t)$ with $\lambda = \Delta t / \Delta x < \lambda_1$ we have the following:*

1. *The following expectation given by $\gamma = \gamma(x_n, t_{l+1}; \xi)$*

$$E_{\mu(\cdot; \xi)} \left[\sum_{0 < k \leq l+1} L^c(\gamma^k, t_{k-1}, \xi_{m(\gamma^k)}^k) \Delta t + v_\Delta^0(\gamma^0) \right] + h(c)t_{l+1}$$

has the infimum with respect to $\xi : G \rightarrow [-\lambda^{-1}, \lambda^{-1}]$ for each fixed $n \in \mathbb{Z}$ and $0 < l+1 \leq k(T)$. The infimum is attained by ξ^ that satisfies $|\xi^*| \leq \lambda_1^{-1} < \lambda^{-1}$.*

2. *The solution v_n^{l+1} of (2.5) satisfies for each $n \in \mathbb{Z}$ and $0 < l+1 \leq k(T)$*

$$v_n^{l+1} = \inf_{\xi} E_{\mu(\cdot; \xi)} \left[\sum_{0 < k \leq l+1} L^c(\gamma^k, t_{k-1}, \xi_{m(\gamma^k)}^k) \Delta t + v_\Delta^0(\gamma^0) \right] + h(c)t_{l+1}.$$

3. *The minimizing velocity field ξ^* for each v_n^{l+1} is unique and satisfies in G*

$$L_\xi^c(x_m, t_k, \xi_m^{*k+1}) = D_x v_{m+1}^k \quad (\Leftrightarrow \xi_m^{*k+1} = H_p(x_m, t_k, c + D_x v_{m+1}^k)).$$

4. *Let ξ^* be the minimizing velocity field for v_n^{l+1} . Let $\gamma = \gamma(x_n, t_{l+1}; \xi^*)$ and $\mu(\cdot; \xi^*)$ be the minimizing random walk and its probability measure. Let $\tilde{\xi}^*$ be the minimizing velocity field for v_{n+2}^{l+1} . Let $\tilde{\gamma} = \gamma(x_{n+2}, t_{l+1}; \tilde{\xi}^*)$ and $\tilde{\mu}(\cdot; \tilde{\xi}^*)$ be the minimizing random walk and its probability measure. Then $u_{n+1}^{l+1} = D_x v_{n+2}^{l+1}$ satisfies*

$$\begin{aligned} u_{n+1}^{l+1} &\leq E_{\mu(\cdot; \xi^*)} \left[\sum_{0 < k \leq l+1} L_x^c(\gamma^k, t_{k-1}, \xi_{m(\gamma^k)}^{*k}) \Delta t + u_\Delta^0(\gamma^0 + \Delta x) \right] + O(\Delta x), \\ u_{n+1}^{l+1} &\geq E_{\tilde{\mu}(\cdot; \tilde{\xi}^*)} \left[\sum_{0 < k \leq l+1} L_x^c(\tilde{\gamma}^k, t_{k-1}, \tilde{\xi}_{m(\tilde{\gamma}^k)}^{*k}) \Delta t + u_\Delta^0(\tilde{\gamma}^0 - \Delta x) \right] + O(\Delta x), \end{aligned}$$

where $O(\Delta x)$ stands for a number of $(-\theta \Delta x, \theta \Delta x)$ with $\theta > 0$ independent of Δx .

Now we take hyperbolic scaling limit:

5. Let v be the viscosity solution of (1.4). Then for each $t \in [0, T]$

$$v_\Delta(\cdot, t) \rightarrow v(\cdot, t) \text{ uniformly on } \mathbb{T} \text{ as } \Delta \rightarrow 0.$$

In particular, we have an error estimate: There exists $\beta_2 > 0$ independent of Δ , $c \in [c_0, c_1]$ and initial data $v^0 \in Lip_r(\mathbb{T})$ such that

$$\sup_{t \in [0, T]} \|v_\Delta(\cdot, t) - v(\cdot, t)\|_{C^0(\mathbb{T})} \leq \beta_2 \sqrt{\Delta x}.$$

6. Let $(x, t) \in \mathbb{T} \times (0, T]$ be a regular point and $\gamma^* : [0, t] \rightarrow \mathbb{R}$ be the minimizing curve for $v(x, t)$. Let (x_n, t_{l+1}) be a point of $[x - 2\Delta x, x + 2\Delta x] \times [t - \Delta t, t + \Delta t]$ and $\gamma_\Delta : [0, t] \rightarrow \mathbb{R}$ be the linear interpolation of the random walk $\gamma = \gamma(x_n, t_{l+1}; \xi^*)$ given by the minimizing velocity field ξ^* for v_n^{l+1} . Then

$$\gamma_\Delta \rightarrow \gamma^* \text{ uniformly on } [0, t] \text{ in probability as } \Delta \rightarrow 0.$$

In particular, the average of γ_Δ converges uniformly to γ^* as $\Delta \rightarrow 0$.

7. Let $u = v_x$ be the entropy solution of (1.3). Then for each regular point $(x, t) \in \mathbb{T} \times [0, T]$

$$u_\Delta(x, t) \rightarrow u(x, t) \text{ as } \Delta \rightarrow 0.$$

In particular, u_Δ converges uniformly to u on $(\mathbb{T} \times [0, T]) \setminus \Theta$, where Θ is a neighborhood of the set of points of discontinuity of u with an arbitrarily small measure.

Note that 1. and 3. give the stability condition of the Lax-Friedrichs scheme, which is called the *CFL-condition*

$$|\lambda H_p(x_m, t_k, c + u_m^k)| < 1.$$

We further state preliminary results on the Lax-Friedrichs scheme. Introduce the solution operators of (2.3) and (2.5)

$$\phi_\Delta^t : L_{r,0}^\infty(\mathbb{T}) \ni u^0 \mapsto u_\Delta(\cdot, t) \in L^\infty(\mathbb{T}), \quad \psi_\Delta^t : Lip_r(\mathbb{T}) \ni v^0 \mapsto v_\Delta(\cdot, t) \in Lip(\mathbb{T}).$$

When we specify the value of c , we write $\phi_\Delta^t(\cdot; c)$, $\psi_\Delta^t(\cdot; c)$, u_Δ^c , $u_m^k(c)$, v_Δ^c , $v_{m+1}^k(c)$. Note that we first get the step function u_Δ^0 from u^0 as (2.4) and then maps u_Δ^0 to $u_\Delta(\cdot, t)$ by ϕ_Δ^t . Similarly we first get the piecewise linear function v_Δ^0 from v^0 as (2.6) which satisfies $u^0 = v_x^0$ and then maps v_Δ^0 to $v_\Delta(\cdot, t)$ by ψ_Δ^t .

Proposition 2.7. Fix $t \in [0, T]$. For each sequence $v_j^0 \rightarrow v^0$ uniformly and $c^j \rightarrow c$ as $j \rightarrow \infty$ ($v_{j,x}^0$ is not necessarily convergent), we have

$$\psi_\Delta^t(v_j^0; c^j) \rightarrow \psi_\Delta^t(v^0; c) \text{ uniformly, } \phi_\Delta^t(v_{j,x}^0; c^j) \rightarrow \phi_\Delta^t(v_x^0; c) \text{ in } L^1(\mathbb{T}) \quad \text{as } j \rightarrow \infty.$$

Proof. It is enough to show $\psi_{\Delta}^{t_{l+1}}(v_j^0; c^j)(x_n) \rightarrow \psi_{\Delta}^{t_{l+1}}(v^0; c)(x_n)$ uniformly with respect to x_n as $j \rightarrow \infty$. Using the stochastic and variational representation, we have

$$\begin{aligned}\psi_{\Delta}^{t_{l+1}}(v^0; c)(x_n) &= E_{\mu(\cdot; \xi^*)} \left[\sum_{0 < k \leq l+1} L(\gamma^k, t_{k-1}, \xi_{m(\gamma^k)}^{*k}) - c \xi_{m(\gamma^k)}^{*k} \Delta t + v_{\Delta}^0(\gamma^0) \right] \\ &\quad + h(c) t_{l+1}, \\ \psi_{\Delta}^{t_{l+1}}(v_j^0; c^j)(x_n) &= E_{\mu(\cdot; \xi_j^*)} \left[\sum_{0 < k \leq l+1} L(\gamma^k, t_{k-1}, \xi_{m(\gamma^k)}^{*k}) - c^j \xi_{m(\gamma^k)}^{*k} \Delta t + v_{j\Delta}^0(\gamma^0) \right] \\ &\quad + h(c^j) t_{l+1},\end{aligned}$$

where ξ^*, ξ_j^* are minimizing velocity field. Hence, by the stochastic and variational representation again, we have

$$\begin{aligned}\psi_{\Delta}^{t_{l+1}}(v_j^0; c^j)(x_n) - \psi_{\Delta}^{t_{l+1}}(v^0; c)(x_n) &\leq E_{\mu(\cdot; \xi^*)} \left[\sum_{0 < k \leq l+1} -(c^j - c) \xi_{m(\gamma^k)}^{*k} \Delta t + v_{j\Delta}^0(\gamma^0) - v_{\Delta}^0(\gamma^0) \right] \\ &\quad + (h(c^j) - h(c)) t_{l+1}, \\ \psi_{\Delta}^{t_{l+1}}(v_j^0; c^j)(x_n) - \psi_{\Delta}^{t_{l+1}}(v^0; c)(x_n) &\geq E_{\mu(\cdot; \xi_j^*)} \left[\sum_{0 < k \leq l+1} -(c^j - c) \xi_{m(\gamma^k)}^{*k} \Delta t + v_{j\Delta}^0(\gamma^0) - v_{\Delta}^0(\gamma^0) \right] \\ &\quad + (h(c^j) - h(c)) t_{l+1}.\end{aligned}$$

Since ξ^*, ξ_j^* are uniformly bounded, we conclude the assertion.

The second convergence follows from the first one and the following relation:

$$\phi_{\Delta}^t(v_{jx}^0; c^j)(x_m) = \frac{\psi_{\Delta}^t(v_j^0; c^j)(x_{m+1}) - \psi_{\Delta}^t(v_j^0; c^j)(x_{m-1})}{2\Delta x}.$$

□

We show details of the one-sided Lipschitz condition of u_m^k which leads to the entropy condition of u . *We remake that the one-sided Lipschitz condition of u_m^k is not necessary for Theorem 2.6, namely u_m^k converges to the exact entropy solution regardless of the condition.* However we want to investigate the one-sided Lipschitz condition in addition to Theorem 2.6, in order to obtain further results. It is easy to prove the one-sided Lipschitz condition through v_{m+1}^k to be a “semiconcave function” of a finite difference version, using the stochastic and variational representation. Unlike the case of exact viscosity solutions, however, we need the assumption of v^0 to be semiconcave. Even if v^0 is semiconcave, it is not easy to estimate time-dependency of the one-sided Lipschitz constant. Therefore we use a direct method similar to that of Lemma 2 in [14]. The direct method is available up to an arbitrary $T > 0$, because we already know about boundedness of difference solutions up to T due to Theorem 2.6. Introduce the following

notation with λ_1 in Theorem 2.6:

$$\begin{aligned}
u^* &:= \sup_{x,t \in \mathbb{T}, c \in [c_0, c_1], |\xi| \leq \lambda_1^{-1}} |L_\xi^c(x, t, \xi)| \quad (\text{note that } |u_m^k| \leq u^*), \\
H_{xx}^* &:= \sup_{x,t \in \mathbb{T}, c \in [c_0, c_1], |u| \leq u^*} |H_{xx}(x, t, c + u)|, \quad H_{xp}^* := \sup_{x,t \in \mathbb{T}, c \in [c_0, c_1], |u| \leq u^*} |H_{xp}(x, t, c + u)|, \\
H_{pp}^* &:= \inf_{x,t \in \mathbb{T}, c \in [c_0, c_1], |u| \leq u^*} |H_{pp}(x, t, c + u)| \quad (H_{pp}^* > 0 \text{ due to (A2)}), \\
\eta &:= \max\{2H_{xp}^* + H_{pp}^*, \frac{1}{2}H_{pp}^* + H_{xx}^*\}, \quad E^* := \frac{2H_{xp}^*}{H_{pp}^*} + \sqrt{4\left(\frac{H_{xp}^*}{H_{pp}^*}\right)^2 + \frac{2H_{xx}^*}{H_{pp}^*}}.
\end{aligned}$$

Proposition 2.8. *Let $\lambda_1 > 0$ be that of Theorem 2.6. Suppose that $\Delta = (\Delta x, \Delta t)$ satisfies $\lambda = \Delta t / \Delta x < \lambda_1$, $\Delta t < \min\{(2\eta)^{-1}, (2E^*H_{pp}^*)^{-1}\}$,*

$$(2.7) \quad \sup_{x,t \in \mathbb{T}, c \in [c_0, c_1], |u| \leq u^*} \lambda(|H_p(x, t, c + u)| + H_{xp}^* \cdot 2\Delta x) < 1, \quad \lambda \leq \frac{1}{rH_{pp}^*}.$$

Then the following holds:

1. We have for $1 \leq k \leq k(T)$

$$E_\Delta^k := \max_m \frac{u_{m+2}^k - u_m^k}{2\Delta x} \leq \frac{2e^{\eta t_k}}{H_{pp}^*} \frac{1}{t_k} \quad (t_k = k\Delta t).$$

2. If $E_\Delta^0 \leq E^*$, we have $E_\Delta^k \leq E^*$ for $1 \leq k \leq k(T)$.

3. If $k > k(\eta^{-1})$, we have $E_\Delta^k \leq \frac{2e^2\eta}{H_{pp}^*}$.

4. If u_m^k is extended to $k \rightarrow \infty$ with $|u_m^k| \leq u^*$, we have $\limsup_{k \rightarrow \infty} E_\Delta^k \leq E^*$.

Proof. Set $z_m^k := u_{m+2}^k - u_m^k$. Using the difference equation and Taylor's formula, we have

$$\begin{aligned}
z_{m+1}^{k+1} &= \frac{z_m^k + z_{m+2}^k}{2} - \frac{\Delta t}{2\Delta x} \{H(x_{m+4}, t_k, c + u_{m+4}^k) - H(x_{m+2}, t_k, c + u_{m+4}^k) \\
&\quad + H(x_{m+2}, t_k, c + u_{m+4}^k) - H(x_{m+2}, t_k, c + u_{m+2}^k) \\
&\quad + H(x_m, t_k, c + u_m^k) - H(x_{m+2}, t_k, c + u_m^k) \\
&\quad + H(x_{m+2}, t_k, c + u_m^k) - H(x_{m+2}, t_k, c + u_{m+2}^k)\} \\
&= \left(\frac{1}{2} + \frac{\lambda}{2}H_p(x_{m+2}, t_k, c + u_{m+2}^k)\right)z_m^k + \left(\frac{1}{2} - \frac{\lambda}{2}H_p(x_{m+2}, t_k, c + u_{m+2}^k)\right)z_{m+2}^k \\
&\quad - \frac{\Delta t}{2\Delta x} \{(H_x(x_{m+2}, t_k, c + u_{m+4}^k) - H_x(x_{m+2}, t_k, c + u_m^k))(2\Delta x) \\
&\quad + \frac{1}{2}H_{pp} \cdot (z_{m+2}^k)^2 + \frac{1}{2}H_{pp} \cdot (z_m^k)^2 + \frac{1}{2}H_{xx} \cdot (2\Delta x)^2 + \frac{1}{2}H_{xx} \cdot (2\Delta x)^2\} \\
&= \left\{\frac{1}{2} + \frac{\lambda}{2}H_p(x_{m+2}, t_k, c + u_{m+2}^k) - \frac{\lambda}{2}H_{xp} \cdot 2\Delta x\right\}z_m^k \\
&\quad + \left\{\frac{1}{2} - \frac{\lambda}{2}H_p(x_{m+2}, t_k, c + u_{m+2}^k) - \frac{\lambda}{2}H_{xp} \cdot 2\Delta x\right\}z_{m+2}^k \\
&\quad - \frac{\Delta t}{2\Delta x} \left\{\frac{1}{2}H_{pp} \cdot (z_{m+2}^k)^2 + \frac{1}{2}H_{pp} \cdot (z_m^k)^2 + \frac{1}{2}H_{xx} \cdot (2\Delta x)^2 + \frac{1}{2}H_{xx} \cdot (2\Delta x)^2\right\}
\end{aligned}$$

By the first inequality in (2.7), it holds that

$$\left\{ \frac{1}{2} \pm \frac{\lambda}{2} H_p(x_{m+2}, t_k, c + u_{m+2}^k) - \frac{\lambda}{2} H_{xp} \cdot 2\Delta x \right\} > 0.$$

Hence, setting $\tilde{z}_m^k := \max\{z_m^k, z_m^k\}$, we obtain

$$\begin{aligned} z_{m+1}^{k+1} &\leq (1 - 2H_{xp}\Delta t)\tilde{z}_m^k + H_{xx}^* \cdot 2\Delta x\Delta t - \frac{H_{pp}^*}{2} \frac{\Delta t}{2\Delta x} (\tilde{z}_m^k)^2, \\ \frac{z_{m+1}^{k+1}}{2\Delta x} &\leq (1 - 2H_{xp}\Delta t) \frac{\tilde{z}_m^k}{2\Delta x} + H_{xx}^* \Delta t - \frac{H_{pp}^*}{2} \Delta t \left(\frac{\tilde{z}_m^k}{2\Delta x} \right)^2. \end{aligned}$$

Note that $g(y) := (1 - 2H_{xp}\Delta t)y + H_{xx}^* \Delta t - (\frac{H_{pp}^*}{2} \Delta t)y^2$ is monotone increasing, if $y \leq (1 - 2H_{xp}\Delta t)/(H_{pp}^* \Delta t)$. It follows from the second inequality in (2.7) and $\Delta t < (2\eta)^{-1}$ that $E_\Delta^0 \leq r/2\Delta x \leq (1 - 2H_{xp}^* \Delta t)/(H_{pp}^* \Delta t) \leq (1 - 2H_{xp}\Delta t)/(H_{pp}^* \Delta t)$ for all initial datas. Suppose that $E_\Delta^k \leq (1 - 2H_{xp}^* \Delta t)/(H_{pp}^* \Delta t)$. Then we obtain

$$(2.8) \quad E_\Delta^{k+1} \leq E_\Delta^k - \Delta t \left\{ \frac{H_{pp}^*}{2} (E_\Delta^k)^2 - 2H_{xp}^* E_\Delta^k - H_{xx}^* \right\}.$$

Since $\Delta t < (2E^* H_{pp}^*)^{-1}$, we have $E^* < (1 - 2H_{xp}^* \Delta t)/(H_{pp}^* \Delta t)$. If $E^* < E_\Delta^k \leq (1 - 2H_{xp}^* \Delta t)/(H_{pp}^* \Delta t)$, we have $E_\Delta^{k+1} < E_\Delta^k$. If $E_\Delta^k \leq E^*$, we have $E_\Delta^{k+1} \geq E_\Delta^k$ but $E_\Delta^{k+1} \leq E^*$. Therefore it holds that $E_\Delta^k \leq (1 - 2H_{xp}^* \Delta t)/(H_{pp}^* \Delta t)$ for all $0 \leq k \leq k(T)$. Thus (2.8) holds for all $0 \leq k \leq k(T)$. We can check that E_Δ^k may increase up to E^* , if $E_\Delta^0 \leq E^*$, and E_Δ^k are bounded by a monotone decreasing sequence with $\limsup_{k \rightarrow \infty} E_\Delta^k \leq E^*$, if $E_\Delta^0 > E^*$.

Now we follow Lemma 2 in [14]. Set $V^k := E_\Delta^k + 1 \geq 1$. Then we have

$$V^{k+1} \leq (1 + \eta\Delta t)V^k - \frac{H_{pp}^*}{2} (V^k)^2$$

Set $W^k := (1 - \eta\Delta t)^k V^k$, $k \geq 0$. Then we have for $k \geq 1$

$$\begin{aligned} W^{k+1} &\leq (1 - \eta\Delta t)(1 + \eta\Delta t)W^k - \frac{H_{pp}^*}{2} \Delta t (W^k)^2 (1 - \eta\Delta t)^{-k+1} \\ &\leq W^k - \frac{H_{pp}^*}{2} \Delta t (W^k)^2. \end{aligned}$$

Consider $w'(t) = -\frac{H_{pp}^*}{2} (w(t))^2$, $w(0) = w^0 := 2/(H_{pp}^* \Delta t)$. The solution satisfies

$$w(t) = \frac{1}{\frac{H_{pp}^*}{2} t + \frac{1}{w^0}} \leq \frac{2}{H_{pp}^* t}.$$

We can show that $W^k \leq w(k\Delta t)$ for $k \geq 1$ as follows: Note that $w(\Delta t) = 1/(H_{pp}^* \Delta t)$ and $W^1 = (1 - \eta\Delta t)(E_\Delta^1 + 1) \leq r/(2\Delta x) + 1$. It follows from $\Delta t < (2\eta)^{-1}$ and the second inequality in (2.7) that $W^1 \leq w(\Delta t)$. Suppose that $W^k \leq w(k\Delta t)$ for some $k \geq 1$. Then, since $g(y) := y - \frac{H_{pp}^* \Delta t}{2} y^2$ is monotone increasing for $y \leq 1/(H_{pp}^* \Delta t)$, $w(k\Delta t) \leq 1/(H_{pp}^* \Delta t)$ and $w'' > 0$, we have

$$\begin{aligned} W^{k+1} &\leq W^k - \frac{H_{pp}^* \Delta t}{2} (W^k)^2 \leq w(k\Delta t) - \frac{H_{pp}^* \Delta t}{2} (w(k\Delta t))^2 \\ &= w(k\Delta t) + \Delta t w'(k\Delta t) = w(k\Delta t + \Delta t) - \frac{1}{2} w'' \cdot (\Delta t)^2 \\ &\leq w((k+1)\Delta t). \end{aligned}$$

Thus we obtain

$$E_{\Delta}^k \leq (1 - \eta\Delta t)^{-k} \frac{2}{H_{pp}^* k \Delta t} \leq (1 - \eta\Delta t)^{-\frac{\eta k \Delta t}{\eta \Delta t}} \frac{2}{H_{pp}^* k \Delta t} \leq \frac{2e^{\eta t_k}}{H_{pp}^*} \frac{1}{t_k}.$$

Set $f(t) := \frac{2e^{\eta t}}{H_{pp}^*} \frac{1}{t}$. The minimum of f is $f(\eta^{-1}) = \frac{2e\eta}{H_{pp}^*}$, which is larger than E^* . Since E_{Δ}^k with $E_{\Delta}^0 > E^*$ are bounded by monotone decreasing numbers, we conclude that $E_{\Delta}^k \leq f(\eta^{-1} + \Delta t) \leq \frac{2e^2\eta}{H_{pp}^*}$ for $k > k(\eta^{-1})$. \square

3 Time Global Stability and Long Time Behavior

We prove time global stability of the Lax-Friedrichs scheme with a fixed mesh size. Then we show long time behaviors of the scheme in which each difference solution fall into a time periodic state with the period 1, obtaining space-time periodic difference solutions. The notion of the effective Hamiltonian of (1.6) arises.

3.1 Time Global Stability

The main result of this section is the following:

Theorem 3.1. *There exists $\lambda_1 > 0$ and $\delta > 0$ such that, if $\Delta = (\Delta x, \Delta t)$ satisfies $0 < \lambda_0 \leq \lambda := \Delta t / \Delta x < \lambda_1$ and $\Delta x \leq \delta$, the Lax-Friedrichs scheme starting from any $u^0 \in L_{r,0}^{\infty}(\mathbb{T})$ works up to an arbitrary time index with the CFL-condition*

$$|H_p(x_m, t_k, c + u_m^k)| \leq \lambda_1^{-1} < \lambda^{-1} \quad \text{for all } m \in \mathbb{Z} \text{ and } k \in \mathbb{Z}_+.$$

In order to prove this theorem, we need uniform boundedness of $\|\phi_{\Delta}^1(u^0; c)\|_{L^{\infty}}$ with respect to $(u^0; c)$ similar to Proposition 2.1. First we observe the following lemma:

Lemma 3.2. *Let $\lambda_1 > 0$ be that of Theorem 2.6 and $\Delta = (\Delta x, \Delta t)$ be such that $0 < \lambda_0 \leq \lambda = \Delta t / \Delta x < \lambda_1$. Fix arbitrarily $t \in (0, T]$. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon; t) > 0$ such that, if $\Delta x \leq \delta$, we have*

$$\sup_{u^0 \in L_{r,0}^{\infty}(\mathbb{T}), c \in [c_0, c_1]} \|\phi_{\Delta}^t(u^0; c) - \phi^t(u^0; c)\|_{L^1(\mathbb{T})} \leq \varepsilon.$$

Proof. If not, for some $\varepsilon_0 > 0$ and $\delta_j \rightarrow 0$, we have $\Delta x_j \leq \delta_j$ such that

$$(3.1) \quad \sup_{u^0 \in L_{r,0}^{\infty}(\mathbb{T}), c \in [c_0, c_1]} \|\phi_{\Delta_j}^t(u^0; c) - \phi^t(u^0; c)\|_{L^1(\mathbb{T})} > \varepsilon_0,$$

where $\Delta_j = (\Delta x_j, \lambda \Delta x_j)$. We show that, for each j , there exists $(u^0, c) \in L_{r,0}^{\infty}(\mathbb{T}) \times [c_0, c_1]$ which attains the supremum (3.1) denoted by b : Let (u_i^0, c^i) be the sequence for which $\|\phi_{\Delta_j}^t(u_i^0; c^i) - \phi^t(u_i^0; c^i)\|_{L^1(\mathbb{T})} \rightarrow b$ as $i \rightarrow \infty$. Let v_i^0 be a primitive of u_i^0 which belongs to $Lip_r(\mathbb{T})$. We have a subsequence of (v_i^0, c^i) , still denoted by (v_i^0, c^i) , which converge to (v^0, c) . By Proposition 2.4 and 2.7, we have $\phi_{\Delta_j}^t(v_i^0; c^i) \rightarrow \phi_{\Delta_j}^t(v_x^0; c)$ in $L^1(\mathbb{T})$ and $\phi^t(v_i^0; c^i) \rightarrow \phi^t(v_x^0; c)$ in $L^1(\mathbb{T})$ as $i \rightarrow \infty$. Therefore we obtain $\|\phi_{\Delta_j}^t(v_x^0; c) - \phi^t(v_x^0; c)\|_{L^1(\mathbb{T})} = b$.

Let (u_j^0, c^j) attain (3.1). Let v_j^0 be a primitive of u_j^0 which belongs to $Lip_r(\mathbb{T})$. We have a subsequence of (v_j^0, c^j) , still denoted by (v_j^0, c^j) , which converges to (v^0, c) . It follows from 7. of Theorem 2.6 that there exists $\delta_0 > 0$ such that, if $\Delta x \leq \delta_0$, we have $\|\phi_\Delta^t(v_x^0; c) - \phi^t(v_x^0; c)\|_{L^1(\mathbb{T})} < \frac{\varepsilon_0}{2}$. Hence

$$\begin{aligned} \frac{\varepsilon_0}{2} &> \|\phi_\Delta^t(v_x^0; c) - \phi^t(v_x^0; c)\|_{L^1(\mathbb{T})} \\ &\geq \|\phi_\Delta^t(v_{jx}^0; c^j) - \phi^t(v_{jx}^0; c^j)\|_{L^1(\mathbb{T})} - \|\phi_\Delta^t(v_x^0; c) - \phi_\Delta^t(v_{jx}^0; c^j)\|_{L^1(\mathbb{T})} \\ &\quad - \|\phi^t(v_{jx}^0; c^j) - \phi^t(v_x^0; c)\|_{L^1(\mathbb{T})}. \end{aligned}$$

By Proposition 2.4 and 2.7, $\|\phi_\Delta^t(v_x^0; c) - \phi_\Delta^t(v_{jx}^0; c^j)\|_{L^1(\mathbb{T})} + \|\phi^t(v_{jx}^0; c^j) - \phi^t(v_x^0; c)\|_{L^1(\mathbb{T})} \leq \frac{\varepsilon_0}{2}$ for large j . Therefore we have $\|\phi_\Delta^t(v_{jx}^0; c^j) - \phi^t(v_{jx}^0; c^j)\|_{L^1(\mathbb{T})} < \varepsilon_0$ for any $\Delta x \leq \delta_0$, which is a contradiction. \square

Next we see that the convergence $\|\phi_\Delta^1(u^0; c) - \phi^1(u^0; c)\|_{L^1(\mathbb{T})} \rightarrow 0$ as $\Delta \rightarrow 0$, which is uniform with respect to (u^0, c) , yields uniform boundedness of $\|\phi_\Delta^1(u^0; c)\|_{L^\infty}$ with the aid of the one-sided Lipschitz condition.

Proposition 3.3. *Let $\lambda_1 > 0$ be that of Theorem 2.6 with $T = 1$ and $\Delta = (\Delta x, \Delta t)$ be such that $0 < \lambda_0 \leq \lambda = \Delta t / \Delta x < \lambda_1$ satisfying the conditions in Proposition 2.8. Then there exists $\delta > 0$ such that, if $\Delta x \leq \delta$, we have with β_1 in Proposition 2.1*

$$\sup_{u^0 \in L_{r,0}^\infty(\mathbb{T}), c \in [c_0, c_1]} \|\phi_\Delta^1(u^0; c)\|_{L^\infty} \leq \beta_1(1) + 1.$$

Proof. Let $0 < \varepsilon < 1$ be such that $\frac{1-\sqrt{\varepsilon}}{3\sqrt{\varepsilon}} > 2e^\eta / H_{pp}^* \geq E_\Delta^{2K}$, where $2K\Delta t = 1$. We have $\delta > 0$ in Lemma 3.2 with this ε and $t = 1$. We take $\Delta x \leq \min\{\delta, \sqrt{\varepsilon}\}$. Consider

$$A := \{y \in \mathbb{T} \mid |\phi_\Delta^1(u^0; c)(y) - \phi^1(u^0; c)(y)| > \sqrt{\varepsilon}\}.$$

Since $\|\phi_\Delta^1(u^0; c) - \phi^1(u^0; c)\|_{L^1(\mathbb{T})} \leq \varepsilon$, $\text{meas}[A] \leq \sqrt{\varepsilon}$. Hence for $y \in A$ there exists $x, \tilde{x} \in \mathbb{R} \setminus A$ such that $0 < y - x \leq \sqrt{\varepsilon}$, $0 < \tilde{x} - y \leq \sqrt{\varepsilon}$. For $x \in \mathbb{T} \setminus A$, we have $|\phi_\Delta^1(u^0; c)(x) - \phi^1(u^0; c)(x)| \leq \sqrt{\varepsilon}$ and $|\phi_\Delta^1(u^0; c)(x)| \leq |\phi^1(u^0; c)(x)| + \sqrt{\varepsilon} \leq \beta_1(t) + 1$. Consider

$$\tilde{A} := \{y \in A \mid |\phi_\Delta^1(u^0; c)(y)| > \beta_1(t) + 1\}.$$

Suppose that \tilde{A} is not empty. Then there exists $x_n \in \tilde{A} \cap (\Delta x \mathbb{Z})$ such that (i) $u_n^{2K} > \beta_1(1) + 1$ or (ii) $u_n^{2K} < -\beta_1(1) - 1$. Since there exist $x_m, x_{m'} \in (\mathbb{R} \setminus A) \cap (\Delta x \mathbb{Z})$ such that $0 < x_n - x_m \leq \sqrt{\varepsilon} + 2\Delta x \leq 3\sqrt{\varepsilon}$, $0 < x_{m'} - x_n \leq \sqrt{\varepsilon} + 2\Delta x \leq 3\sqrt{\varepsilon}$, we have

$$\begin{aligned} \text{(i)} \quad & \frac{u_n^{2K} - u_m^{2K}}{x_n - x_m} > \frac{\beta_1(1) + 1 - (\beta_1(1) + \sqrt{\varepsilon})}{3\sqrt{\varepsilon}} = \frac{1 - \sqrt{\varepsilon}}{3\sqrt{\varepsilon}} > E_\Delta^{2K}, \\ \text{(ii)} \quad & \frac{u_{m'}^{2K} - u_n^{2K}}{x_{m'} - x_n} > \frac{-(\beta_1(1) + \sqrt{\varepsilon}) - (-\beta_1(1) - 1)}{3\sqrt{\varepsilon}} = \frac{1 - \sqrt{\varepsilon}}{3\sqrt{\varepsilon}} > E_\Delta^{2K}. \end{aligned}$$

These two inequalities contradict the one-sided Lipschitz condition. \square

Remark. 7. of Theorem 2.6 states that $\phi_\Delta^1(u^0; c)$ converges to $\phi^1(u^0; c)$ uniformly on $\mathbb{T} \setminus \Theta$ as $\Delta \rightarrow 0$, where Θ is an arbitrary “small” neighborhood of shocks. We may not use this fact for Proposition 3.3, because it is not easy to obtain uniformness of the convergence with respect to (u^0, c) .

Proof of Theorem 3.1. Let $\lambda_1 > 0$ be that of Theorem 2.6 with $T = 1$ and $r \geq \beta_1(1)+1$. Let $\delta > 0$ be that of Proposition 3.3. Then, for any $u^0 \in L_{r,0}^\infty(\mathbb{T})$, $\tilde{u}^0 := \phi_\Delta^1(u^0; c)$ belongs to $L_{\beta_1(1)+1,0}^\infty(\mathbb{T})$. Hence, by the choice of λ_1 , $\phi_\Delta^1(\tilde{u}^0; c) = \phi_\Delta^2(u^0; c)$ is well-defined and bounded by $\beta_1(1) + 1$ again. In this way $\phi_\Delta^t(u^0; c)$ can be defined for $t \rightarrow \infty$ with the CFL-condition. \square

3.2 Long Time Behavior

If we take $r \geq \beta_1(1) + 1$, $\phi_\Delta^1(u^0; c)$ belongs to $L_{r,0}^\infty(\mathbb{T})$. Therefore $\phi_\Delta^1(\cdot; c)$ is a map from $L_{r,0}^\infty(\mathbb{T})$ into itself. We can find fixed points of the map for each c . In this subsection, we consider the fixed points and their stability, which makes clear the long time behavior of the Lax-Friedrichs scheme. Note that the Lax-Friedrichs scheme has a contraction property under the CFL-condition: For $0 \leq t \leq t'$ we have

$$\| \phi_\Delta^{t'}(u^0; c) - \phi_\Delta^{t'}(\tilde{u}^0; c) \|_{L^1(\mathbb{T})} \leq \| \phi_\Delta^t(u^0; c) - \phi_\Delta^t(\tilde{u}^0; c) \|_{L^1(\mathbb{T})}.$$

This can be sharpened to be a strict contraction property. Let $\sum_{m,k}$ be summation with respect to $\{m \mid 0 \leq m < 2N, m+k = \text{even}\}$ for each fixed k and $\|x\|_1 := \sum_{1 \leq j \leq n} |x_j|$ for $x \in \mathbb{R}^n$.

Proposition 3.4. *The family of maps $\{\phi_\Delta^t(\cdot; c)\}_{t \geq 0}$ has a strict contraction property within the time period 1: For any two different initial datas u^0 and \tilde{u}^0 , we have*

$$\| \phi_\Delta^{t+1}(u^0; c) - \phi_\Delta^{t+1}(\tilde{u}^0; c) \|_{L^1(\mathbb{T})} < \| \phi_\Delta^t(u^0; c) - \phi_\Delta^t(\tilde{u}^0; c) \|_{L^1(\mathbb{T})}.$$

Proof. It is enough to show that any two difference solutions u^k, \tilde{u}^k of (2.3) satisfy for all $k \geq 0$

$$\| u^{k+2K} - \tilde{u}^{k+2K} \|_1 < \| u^k - \tilde{u}^k \|_1.$$

Set $z_m^k := u_m^k - \tilde{u}_m^k$, $\sigma_m^k := \text{sign } z_m^k = 1$ or -1 ($\text{sign } 0 := 1$). Then $\|u^k - \tilde{u}^k\|_1 = \sum_{m,k} |z_m^k| = \sum_{m,k} \sigma_m^k z_m^k$. By the difference equation of (2.3), we have

$$\sum_{m,k} |z_{m+1}^{k+1}| = \sum_{m,k} \sigma_{m+1}^{k+1} z_{m+1}^{k+1} = \sum_{m,k} \sigma_{m+1}^{k+1} \left\{ \frac{1}{2} z_{m+2}^k (1 - \lambda \delta_{m+2}^k) + \frac{1}{2} z_m^k (1 + \lambda \delta_m^k) \right\},$$

where $\delta_m^k := H_p(x_m, t_k, c + u_m^k + \theta_m^k)$ with a constant θ_m^k derived from Taylor's formula. Shifting the order of the above sum, we obtain

$$\begin{aligned} \sum_{m,k} |z_{m+1}^{k+1}| &= \sum_{m,k} z_m^k \left\{ \frac{1}{2} \sigma_{m-1}^{k+1} (1 - \lambda \delta_m^k) + \frac{1}{2} \sigma_{m+1}^{k+1} (1 + \lambda \delta_m^k) \right\} \\ &= \sum_{m,k} |z_m^k| + \sum_{m,k} |z_m^k| \left[-1 + \sigma_m^k \left\{ \frac{1}{2} \sigma_{m-1}^{k+1} (1 - \lambda \delta_m^k) + \frac{1}{2} \sigma_{m+1}^{k+1} (1 + \lambda \delta_m^k) \right\} \right]. \end{aligned}$$

Let R^k denote the second sum in the second line of the above equality. We see that $R^k \leq 0$, since each term of $[\]$ in R^k satisfies

- (1) If $\sigma_{m-1}^{k+1} + \sigma_{m+1}^{k+1} = \pm 2$, then $[\] = -1 \pm \sigma_m^k = 0$ or -2 ,
- (2) If $\sigma_{m-1}^{k+1} + \sigma_{m+1}^{k+1} = 0$, then $[\] = -1 \pm \lambda \delta_m^k < 0$, due to the CFL-condition.

Since u^k, \tilde{u}^k have the same average equal to 0 and $u^0 \neq \tilde{u}^0$, z_m^k necessarily change the sign and the case (2) happens. It seems possible that even if u^k, \tilde{u}^k are as such, we have $R^k = 0$, namely $z_m^k = 0$ for all the integers m for which the case (2) happens. But after some more k^* -time evolution ($k^* < N < 2K$) the case (2) certainly happens and $R^{k+k^*} < 0$, because such zero-points disappear as k increases due to the monotonicity of the Lax-Friedrichs scheme under the CFL-condition (see also Remark 3.4 in [13]). \square

We now show the existence of time periodic difference solutions and their stability, which provides long time behavior of the Lax-Friedrichs scheme.

Theorem 3.5. *Take $r \geq \beta_1(1) + 1$ and fix $\Delta = (\Delta x, \Delta t)$ so that Theorem 2.6 and 3.1 holds. Then, for each c , there exists a fixed point $\bar{u}_\Delta^0 \in L_{r,0}^\infty(\mathbb{T})$ of $\phi_\Delta^1(\cdot; c)$, which yields a time periodic difference solution $\phi_\Delta^t(\bar{u}_\Delta^0; c)$. Such a periodic solution is unique with respect to c . Any other solution $\phi_\Delta^t(u^0; c)$ exponentially falls into the periodic state, namely there exists $\rho \in (0, 1)$ and $a > 0$ depending on Δ , but independent of u^0 , such that $\|\phi_\Delta^t(u^0; c) - \phi_\Delta^t(\bar{u}_\Delta^0; c)\|_{L^\infty} \leq a\rho^t$ for $t \in \mathbb{N}$.*

Proof. The map $\phi_\Delta^1(\cdot; c)$ is actually a map from \mathbb{R}^N to \mathbb{R}^N , since the step functions have only N different values at most. Let B_r be the set of all $x \in \mathbb{R}^N$ with $\|x\|_\infty \leq r$. If $r \geq \beta_1(1) + 1$, then the map $\phi_\Delta^1(\cdot; c)$ is actually a map from B_r to B_r . Therefore we obtain a fixed point due to a fixed point theorem. By Proposition 3.4, periodic solutions must be unique. Exponential decay can be proved in the same way as the proof of (5) of Theorem 2.1 in [13]. \square

Remark. It is likely in general that ρ gets arbitrary close to 1 as Δ tends to 0.

Theorem 3.6. *Take $r \geq \beta_1(1) + 1$ and fix $\Delta = (\Delta x, \Delta t)$ so that Theorem 2.6 and 3.1 holds. Then, for each c , there exists a constant $\bar{h}_\Delta(c) \in \mathbb{R}$ such that if $h(c) = \bar{h}_\Delta(c)$ we have a fixed point $\bar{v}_\Delta^0 \in \text{Lip}_r(\mathbb{T})$ of $\psi_\Delta^1(\cdot; c)$, which yields a time periodic difference solution $\psi_\Delta^t(\bar{v}_\Delta^0; c)$. Such a periodic solution is unique with respect to c up to constants. Any other solution $\psi_\Delta^t(v^0; c)$ exponentially falls into a periodic state, namely for the same $\rho \in (0, 1)$ and $a > 0$ in Theorem 3.5 and $d \in \mathbb{R}$ depending on $(v^0; c)$ we have $\|\psi_\Delta^t(v^0; c) - \psi_\Delta^t(\bar{v}_\Delta^0 + dI_1; c)\|_{C^0} \leq a\rho^t$ for $t \in \mathbb{N}$, where $I_1(x) := 1$ and $\psi_\Delta^t(v^0 + dI_1; c) = \psi_\Delta^t(v^0; c) + dI_1$.*

Proof. We imitate the proof of the weak KAM theorem [8]. The map $\psi_\Delta^1(\cdot; c)$ is actually a map from \mathbb{R}^N to \mathbb{R}^N , since the piecewise linear functions have only N different slopes at most. Let us write $x \sim y$ for $x, y \in \mathbb{R}^N$, if there exists $b \in \mathbb{R}$ such that $y = x + b\mathbf{1}$ with $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}^N$. Introduce $\hat{x} := \{y \in \mathbb{R}^N \mid y \sim x\}$, $\|\hat{x}\| := \inf_{y \in \hat{x}} \|y\|_\infty$, $\hat{B}_r := B_r / \sim$ and

$$\hat{\psi}_\Delta^1(\hat{w}; c) := \hat{y},$$

where $w, y \in \mathbb{R}^N$ are the vectors given as $w_i := \psi_\Delta^0(v^0; c)(x_{2i-1})$, $y_i := \psi_\Delta^1(v^0; c)(x_{2i-1})$, $i = 1, 2, 3, \dots, N$. Note that $\hat{\psi}_\Delta^1(\cdot; c)$ is actually a map from \hat{B}_r into \hat{B}_r . Hence we have a fixed point \hat{w} satisfying $\hat{\psi}_\Delta^1(\hat{w}; c) = \hat{w}$. Therefore, for the linear interpolation of an element $\bar{w} \in \hat{w}$ denoted by \bar{v}_Δ^0 , we have a constant $b(c) \in \mathbb{R}$ such that

$$\bar{v}_\Delta^0 = \psi_\Delta^1(\bar{v}_\Delta^0; c) + b(c)I_1.$$

This relation means that \bar{v}_Δ^0 yields a time periodic solution of (2.5) with $h(c) + b(c)$ instead of $h(c)$.

Note that $\psi_\Delta^t(v^0; c) \leq \psi_\Delta^t(\tilde{v}^0; c)$, if $v^0 \leq \tilde{v}^0$. Let a^0, b^0 be constants such that for all $x \in \mathbb{T}$

$$\bar{v}_\Delta^0(x) + b^0 \leq v^0(x) \leq \bar{v}_\Delta^0(x) + a^0$$

with at least one point attaining the equality in each inequality. Then we have $\bar{v}_\Delta^0(x) + b^0 \leq \psi_\Delta^1(v^0; c)(x) \leq \bar{v}_\Delta^0(x) + a^0$ for all $x \in \mathbb{T}$. Let a^1, b^1 be constants such that for all $x \in \mathbb{T}$

$$\bar{v}_\Delta^0(x) + b^1 \leq \psi_\Delta^1(v^0; c)(x) \leq \bar{v}_\Delta^0(x) + a^1$$

with at least one point attaining the equality in each inequality. Note that $a^1 \leq a^0$ and $b^1 \geq b^0$. Then we have $\bar{v}_\Delta^0(x) + b^1 \leq \psi_\Delta^2(v^0; c)(x) \leq \bar{v}_\Delta^0(x) + a^1$ for all $x \in \mathbb{T}$. In this way we obtain the monotone bounded sequences a^j and b^j . Take d such that $\lim_{j \rightarrow \infty} b^j \leq d \leq \lim_{j \rightarrow \infty} a^j$. Then $\bar{v}_\Delta^0 + dI_1$ and $\psi_\Delta^t(v^0; c)$ coincide at least one point for any $t \in \mathbb{N}$. Let $x_0 \in \mathbb{T}$ be such that $\bar{v}_\Delta^0(x_0) + d = \psi_\Delta^t(v^0; c)(x_0)$. Then we obtain for all $x \in \mathbb{T}$ and $t \in \mathbb{N}$

$$|\psi_\Delta^t(v^0; c)(x) - \psi_\Delta^t(\bar{v}_\Delta^0 + dI_1; c)(x)| \leq \left| \int_{x_0}^x |\phi_\Delta^t(v_x^0; c) - \phi_\Delta^t(\bar{u}_\Delta^0; c)| dy \right| \leq \rho^t.$$

□

Introduce the map $\bar{h}_\Delta(c) : c \mapsto h(c) + b(c)$, which is the effective Hamiltonian of the difference Hamilton-Jacobi equation (1.6). We remark that $\bar{h}_\Delta(c)$ may play an important role in numerical analysis of the weak KAM theory. It is meaningful to investigate its properties.

3.3 Effective Hamiltonian

Here is the characterization of $\bar{h}_\Delta(c)$, which is very similar to that of the effective Hamiltonian $\bar{h}(c)$ of the exact Hamilton-Jacobi equations (1.2). We refer to [1] for the characterization of $\bar{h}(c)$.

Theorem 3.7. *1. $h(c) = \bar{h}_\Delta(c)$ is the unique value for which (1.6) admits a space-time periodic difference solution.*

2. $\bar{h}_\Delta(c)$ is the averaged Hamiltonian: For the space-time periodic difference solution \bar{u}_m^k of (1.5), we have

$$\bar{h}_\Delta(c) = \sum_{0 \leq k < 2K} \sum_{m; k} H(x_m, t_k, c + \bar{u}_m^k(c)) \cdot 2\Delta x \Delta t.$$

3. Let $v_n^{l+1}(c)$ be a time global solution of the difference equation

$$(3.2) \quad D_t v_m^{k+1} + H(x_m, t_k, c + D_x v_{m+1}^k) = 0.$$

Then we have for all n

$$\lim_{l \rightarrow \infty} \frac{v_n^{l+1}(c)}{t_{l+1}} = -\bar{h}_\Delta(c).$$

4. $\bar{h}_\Delta(c)$ is a convex C^1 -function.

5. $\bar{h}_\Delta(c)$ uniformly converges to the exact effective Hamiltonian $\bar{h}(c)$ of (1.2) as $\Delta \rightarrow 0$:

$$\sup_{c \in [c_0, c_1]} |\bar{h}_\Delta(c) - \bar{h}(c)| \leq \beta_3 \sqrt{\Delta x}.$$

Proof. 1. Let \tilde{v}_{m+1}^k be another space-time periodic solution of (1.6) with $h(c) = \tilde{h}_\Delta(c)$. Extending periodic solutions to the entire odd-grid, we have the stochastic and variational representation formula up to any negative time index l_0 :

$$\begin{aligned} \bar{v}_n^{l+1} &= E_{\mu(\cdot; \xi^*)} \left[\sum_{l_0 < k \leq l+1} L^c(\gamma^k, t_{k-1}, \xi_{m(\gamma^k)}^{*k}) \Delta t + \bar{v}_{m(\gamma^{l_0})}^{l_0} \right] + \bar{h}_\Delta(c)(t_{l+1} - t_{l_0}), \\ \tilde{v}_n^{l+1} &= E_{\mu(\cdot; \tilde{\xi}^*)} \left[\sum_{l_0 < k \leq l+1} L^c(\gamma^k, t_{k-1}, \tilde{\xi}_{m(\gamma^k)}^{*k}) \Delta t + \tilde{v}_{m(\gamma^{l_0})}^{l_0} \right] + \tilde{h}_\Delta(c)(t_{l+1} - t_{l_0}). \end{aligned}$$

By the variational property, we have

$$(3.3) \quad \tilde{v}_n^{l+1} - \bar{v}_n^{l+1} \leq E_{\mu(\cdot; \xi^*)} \left[\tilde{v}_{m(\gamma^{l_0})}^{l_0} - \bar{v}_{m(\gamma^{l_0})}^{l_0} \right] + (\tilde{h}_\Delta(c) - \bar{h}_\Delta(c))(t_{l+1} - t_{l_0}).$$

Noting that $\bar{v}_{m+1}^k, \tilde{v}_{m+1}^k$ are periodic and hence bounded, dividing (3.3) by $t_{l+1} - t_{l_0}$ and letting $l_0 \rightarrow -\infty$, we obtain

$$0 \leq \tilde{h}_\Delta(c) - \bar{h}_\Delta(c).$$

A similar reasoning yields the converse inequality and concludes $\tilde{h}_\Delta(c) = \bar{h}_\Delta(c)$.

2. Since \bar{v}_{m+1}^k satisfies $D_t \bar{v}_m^{k+1} + H(x_m, t_k, c + D_x \bar{v}_{m+1}^k) = \bar{h}_\Delta(c)$, we have

$$\bar{h}_\Delta(c) = \sum_{0 \leq k < 2K} \sum_{m; k} D_t \bar{v}_m^{k+1} \cdot 2\Delta x \Delta t + \sum_{0 \leq k < 2K} \sum_{m; k} H(x_m, t_k, c + D_x \bar{v}_{m+1}^k) \cdot 2\Delta x \Delta t.$$

The first term of the right hand side is equal to 0 due to periodicity of \bar{v}_{m+1}^k .

3. Let $\tilde{v}_n^{l+1}(c)$ be the solution of $D_t \tilde{v}_m^{k+1} + H(x_m, t_k, c + D_x \tilde{v}_{m+1}^k) = \bar{h}_\Delta(c)$ with the same mesh size as (3.2) and $\tilde{v}_{m+1}^0 = v_{m+1}^0$. It follows from Theorem 3.6 that we have, adding a constant if necessary, $|\tilde{v}_n^{l+1}(c) - \bar{v}_n^{l+1}(c)| \rightarrow 0$ as $l \rightarrow \infty$. Therefore \tilde{v}_n^{l+1} is bounded for $l \rightarrow \infty$. Since

$$\begin{aligned} v_n^{l+1}(c) &= \inf_{\xi} E_{\mu(\cdot; \xi)} \left[\sum_{0 < k \leq l+1} L^c(\gamma^k, t_{k-1}, \xi_{m(\gamma^k)}^k) \Delta t + v_{m(\gamma^0)}^0 \right], \\ \tilde{v}_n^{l+1}(c) &= \inf_{\xi} E_{\mu(\cdot; \xi)} \left[\sum_{0 < k \leq l+1} L^c(\gamma^k, t_{k-1}, \xi_{m(\gamma^k)}^k) \Delta t + v_{m(\gamma^0)}^0 \right] + \bar{h}_\Delta(c)t_{l+1} \end{aligned}$$

and their minimizing velocity fields are the same, we obtain $v_n^{l+1}(c) - \tilde{v}_n^{l+1}(c) = -\bar{h}_\Delta(c)t_{l+1}$.

4. Following the proof of (6) of Theorem 2.1 in [13], we can prove that $c + \bar{u}_m^k(c)$ is a C^1 -function for each m, k . Therefore 2. yields C^1 -regularity of \bar{h}_Δ . Let $v_n^{l+1}(c)$ be a solution of (3.2). Fix n . The map $c \mapsto v_n^{l+1}(c)$ is a concave function for each $l+1 \geq 1$:

Let ξ^* be the minimizing velocity field for $v_n^{l+1}(c^*)$ with $c^* := \lambda c + (1 - \lambda)\tilde{c}$, $\lambda \in [0, 1]$. Then we have

$$\begin{aligned} & v_n^{l+1}(c^*) - \{\lambda v_n^{l+1}(c) + (1 - \lambda)v_n^{l+1}(\tilde{c})\} \\ & \geq \lambda E_{\mu(\cdot; \xi^*)} \left[\sum_{0 < k \leq l+1} -(c^* - c) \xi_{m(\gamma^k)}^{*k} \Delta t \right] + (1 - \lambda) E_{\mu(\cdot; \xi^*)} \left[\sum_{0 < k \leq l+1} -(c^* - \tilde{c}) \xi_{m(\gamma^k)}^{*k} \Delta t \right] \\ & = 0. \end{aligned}$$

Therefore the map $c \mapsto v_n^{l+1}(c)/t_{l+1}$ is also a concave function and $\bar{h}_\Delta(c) = -\lim_{l \rightarrow \infty} v_n^{l+1}(c)/t_{l+1}$ is a convex function.

5. Hereafter b_1, b_2, \dots are positive constants independent of Δ and c . For each $x \in \mathbb{T}$, we have m such that $x \in [x_{m+1}, x_{m+3})$. Note that $\bar{v}_{n_*+1}^{2K} \leq \bar{v}_\Delta(x, 1) \leq \bar{v}_{n^*+1}^{2K}$ with $(n_*, n^*) = (m, m+2)$ or $(m+2, m)$ and

$$(3.4) \quad \bar{v}_{n_*+1}^{2K} - \bar{v}(x_{n_*+1}, 1) - 2r\Delta x \leq \bar{v}_\Delta(x, 1) - \bar{v}(x, 1) \leq \bar{v}_{n^*+1}^{2K} - \bar{v}(x_{n^*+1}, 1) + 2r\Delta x.$$

Let $x \in \mathbb{T}$ attain $\max_{y \in \mathbb{T}} (\bar{v}_\Delta(y, 1) - \bar{v}(y, 1))$ and n^* be defined in the above way with the x . Let γ^* be a minimizing curve for $\bar{v}(x_{n^*+1}, t)$. Define ξ as $\xi_m^k := \gamma^{*'}(t_k)$. Note that $\eta^k(\gamma)$ defined by the ξ satisfies $|\eta^k(\gamma) - \gamma^*(t_k)| \leq b_1\Delta x$ for any $0 \leq k \leq 2K$. By the representation formulas and Proposition 2.5, we have

$$\begin{aligned} \bar{v}(x_{n^*+1}, 1) &= \int_0^1 L^c(\gamma^*(s), s, \gamma^{*'}(s)) ds + \bar{v}(\gamma^*(0), 0) + \bar{h}(c), \\ (3.5) \quad \bar{v}_{n^*+1}^{2K} &\leq E_{\mu(\cdot; \xi)} \left[\sum_{0 < k \leq 2K} L^c(\gamma^k, t_{k-1}, \xi_{m(\gamma^k)}^k) \Delta t + \bar{v}_\Delta(\gamma^0, 0) \right] + \bar{h}_\Delta(c) \\ &\leq E_{\mu(\cdot; \xi)} \left[\sum_{0 < k \leq 2K} L^c(\eta^k(\gamma), t_{k-1}, \xi_{m(\gamma^k)}^k) \Delta t + \bar{v}_\Delta(\eta^0(\gamma), 0) \right] + \bar{h}_\Delta(c) + b_2\sqrt{\Delta x} \\ &\leq E_{\mu(\cdot; \xi)} \left[\sum_{0 < k \leq 2K} L^c(\gamma^*(t_k), t_{k-1}, \gamma^{*'}(t_k)) \Delta t + \bar{v}_\Delta(\gamma^*(0), 0) \right] + \bar{h}_\Delta(c) + b_3\sqrt{\Delta x} \\ &\leq \int_0^1 L^c(\gamma^*(s), s, \gamma^{*'}(s)) ds + \bar{v}_\Delta(\gamma^*(0), 0) + \bar{h}_\Delta(c) + b_4\sqrt{\Delta x}. \end{aligned}$$

Therefore, noting (3.4), we have

$$\bar{v}_\Delta(x, 1) - \bar{v}(x, 1) \leq \bar{v}_\Delta(\gamma^*(0), 0) - \bar{v}(\gamma^*(0), 0) + \bar{h}_\Delta(c) - \bar{h}(c) + b_5\sqrt{\Delta x}.$$

It follows from the periodicity of \bar{v}_Δ, \bar{v} and the choice of the x that $(\bar{v}_\Delta(x, 1) - \bar{v}(x, 1)) - (\bar{v}_\Delta(\gamma^*(0), 0) - \bar{v}(\gamma^*(0), 0)) \geq 0$. Therefore we obtain

$$-b_5\sqrt{\Delta x} \leq \bar{h}_\Delta(c) - \bar{h}(c).$$

Let $x \in \mathbb{T}$ attain $\min_{y \in \mathbb{T}} (\bar{v}_\Delta(y, 1) - \bar{v}(y, 1))$ and n_* be defined in the above way with the x . Let ξ^* be the minimizing velocity field for $\bar{v}_{n_*+1}^{2K}$. Then we have

$$\begin{aligned} \bar{v}_{n_*+1}^{2K} &= E_{\mu(\cdot; \xi^*)} \left[\sum_{0 < k \leq 2K} L^c(\gamma^k, t_{k-1}, \xi_{m(\gamma^k)}^{*k}) \Delta t + \bar{v}_\Delta(\gamma^0, 0) \right] + \bar{h}_\Delta(c), \\ &\geq E_{\mu(\cdot; \xi^*)} \left[\sum_{0 < k \leq 2K} L^c(\eta^k(\gamma), t_{k-1}, \xi_{m(\gamma^k)}^{*k}) \Delta t + \bar{v}_\Delta(\eta^0(\gamma), 0) \right] + \bar{h}_\Delta(c) - b_6\sqrt{\Delta x}. \end{aligned}$$

Let $\eta_\Delta(\gamma)$ be the linear interpolation of $\eta^k(\gamma)$. Note that $\eta_\Delta(\gamma)'(t) = \xi_{m(\gamma^k)}^{*k}$ for $t \in (t_{k-1}, t_k)$. For each γ we have

$$(3.6) \quad \begin{aligned} \bar{v}(x_{n_*+1}, 1) &\leq \int_0^1 L^c(\eta_\Delta(\gamma)(s), s, \eta_\Delta(\gamma)'(s)) ds + \bar{v}(\eta_\Delta(\gamma)(0), 0) + \bar{h}(c) \\ &\leq \sum_{0 < k \leq 2K} L^c(\eta^k(\gamma), t_{k-1}, \xi_{m(\gamma^k)}^{*k}) \Delta t + \bar{v}(\eta^0(\gamma), 0) + \bar{h}(c) + b_7 \Delta x. \end{aligned}$$

Therefore, noting (3.4), we have

$$\bar{v}_\Delta(x, 1) - \bar{v}(x, 1) \geq E_{\mu(\cdot; \xi^*)} \left[\bar{v}_\Delta(\eta^0(\gamma), 0) - \bar{v}(\eta^0(\gamma), 0) \right] + \bar{h}_\Delta(c) - \bar{h}(c) - b_8 \sqrt{\Delta x}.$$

Since $(\bar{v}_\Delta(x, 1) - \bar{v}(x, 1)) - (\bar{v}_\Delta(\eta^0(\gamma), 0) - \bar{v}(\eta^0(\gamma), 0)) \leq 0$ for all γ , we obtain

$$\bar{h}_\Delta(c) - \bar{h}(c) \leq b_8 \sqrt{\Delta x}.$$

□

3.4 Convergence of Periodic Solution

It is easy to prove convergence of space-time periodic difference solutions to exact ones up to subsequence. We should remark that space-time periodic viscosity solutions and entropy solutions are not unique with respect to c in general. The selection problem in finite difference approximation remains open. It is also challenging to investigate details of the convergence even in the case where the uniqueness holds. We will make some progress in this issue in the next section.

Theorem 3.8. *There is a sequence $\Delta = (\Delta x, \Delta t) \rightarrow 0$ for which $\{\bar{v}_\Delta^c\}$ and $\{\bar{u}_\Delta^c\}$ converge to a \mathbb{Z}^2 -periodic viscosity solution \bar{v} of (1.2) with $h(c) = \bar{h}(c)$ and a \mathbb{Z}^2 -periodic entropy solution $\bar{u} = \bar{v}_x$ of (1.1) respectively:*

$$\sup_{t \in \mathbb{T}} \|\bar{v}_\Delta^c(\cdot, t) - \bar{v}(\cdot, t)\|_{C^0} \rightarrow 0, \quad \sup_{t \in \mathbb{T}} \|\bar{u}_\Delta^c(\cdot, t) - \bar{u}(\cdot, t)\|_{L^1(\mathbb{T})} \rightarrow 0.$$

Proof. Since $\{\bar{v}_\Delta^c(\cdot, 0)\}$ is a family of Lipschitz functions with a common Lipschitz constant, we have convergent subsequence, still denoted by $\bar{v}_\Delta^c(\cdot, 0)$: $\bar{v}_\Delta^c(\cdot, 0) \rightarrow \bar{v}^0$. Let \bar{v} be the viscosity solution of (1.4) with $v^0 = \bar{v}^0$ and $h(c) = \bar{h}(c)$. Then we have a minimizing curve such that

$$\bar{v}(x_n, t_{l+1}) = \int_0^{t_{l+1}} L^c(\gamma^*(s), s, \gamma^{*'}(s)) ds + \bar{v}^0(\gamma^*(0)) + \bar{h}(c)t_{l+1}.$$

By an estimate similar to (3.5), we have

$$\bar{v}_\Delta(x_n, t_{l+1}) \leq \int_0^{t_{l+1}} L^c(\gamma^*(s), s, \gamma^{*'}(s)) ds + \bar{v}_\Delta(\gamma^*(0), 0) + \bar{h}_\Delta(c)t_{l+1} + b_1 \sqrt{\Delta x}.$$

Since $\bar{h}_\Delta(c) \rightarrow \bar{h}(c)$, we obtain

$$\limsup_{\Delta \rightarrow 0} \{\bar{v}_\Delta^c(x_n, t_{l+1}) - \bar{v}(x_n, t_{l+1})\} \leq 0,$$

which is uniform with respect to $(x_n, t_{l+1}) \in \mathbb{T}^2$. By an estimate similar to (3.6), we obtain

$$\liminf_{\Delta \rightarrow 0} \{\bar{v}_\Delta^c(x_n, t_{l+1}) - \bar{v}(x_n, t_{l+1})\} \geq 0,$$

which is uniform with respect to $(x_n, t_{l+1}) \in \mathbb{T}^2$. Therefore we conclude that $\bar{v}_\Delta^c \rightarrow \bar{v}$ uniformly on \mathbb{T}^2 and \bar{v} is \mathbb{Z}^2 -periodic due to periodicity of \bar{v}_Δ .

It follows from reasoning similar to the proof of Theorem 2.8 in [18] that $\bar{u}_\Delta^c := (\bar{v}_\Delta^c)_x$ converges to $\bar{u} = \bar{v}_x$ pointwise almost everywhere in \mathbb{T}^2 , where $\{\bar{v}_\Delta^c\}$ is the above convergent subsequence. Hence we have $\|\bar{u}_\Delta^c(\cdot, t) - \bar{u}(\cdot, t)\|_{L^1(\mathbb{T})} \rightarrow 0$ for each t . It follows from reasoning similar to the proof of Proposition 2.14 in [13] that \bar{u}_Δ^c satisfies

$$\|\bar{u}_\Delta^c(\cdot, t_k) - \bar{u}_\Delta^c(\cdot, t_{k'})\|_{L^1(\mathbb{T})} \leq b_2 |t_k - t_{k'}|$$

with a constant b_2 independent of k, k', c and Δ . Therefore $\bar{u} \in Lip(\mathbb{T}; L^1(\mathbb{T}))$ with the Lipschitz constant b_2 . Thus we conclude the theorem. \square

4 Error Estimate

We show error estimates for entropy solutions of initial value problems and for \mathbb{Z}^2 -periodic entropy solutions in the special case where they are associated with KAM tori. The latter is a rigorous result on finite difference approximation of KAM tori. We refer to [2] for an error estimate for \mathbb{Z}^2 -periodic entropy solutions associated with KAM tori in the vanishing viscosity method.

4.1 Error Estimate for initial value problem

The following error estimates hold:

Theorem 4.1. *Let $\Delta = (\Delta x, \Delta t)$ satisfy the conditions in Theorem 2.6 and Proposition 2.8. Let u be the entropy solution of (1.3) and u_Δ be given by the difference solution of (2.3). Then the following holds:*

1. *For any $T \in (0, \infty)$ and each $t \in (0, T]$, there exists a constant $\beta_4(t) > 0$ independent of initial data for which we have*

$$\|u_\Delta(\cdot, t) - u(\cdot, t)\|_{L^1(\mathbb{T})} \leq \beta_4(t) \Delta x^{\frac{1}{4}}.$$

In particular, if u^0 satisfies the one-sided Lipschitz condition $u^0(x) - u^0(y) \leq M(x - y)$ for a.e. $x \geq y$ and $M \geq 0$, then there exists a constant $\beta_5 > 0$ for which we have

$$\sup_{0 \leq t \leq T} \|u_\Delta(\cdot, t) - u(\cdot, t)\|_{L^1(\mathbb{T})} \leq \beta_5 \Delta x^{\frac{1}{4}}.$$

2. *If u is Lipschitz in $\mathbb{T} \times [0, T]$, then there exists a constant $\beta_6 > 0$ for which we have*

$$\sup_{(x,t) \in \mathbb{T} \times [0,T]} |u_\Delta(x, t) - u(x, t)| \leq \beta_6 \Delta x^{\frac{1}{4}}$$

Proof. 1. By Theorem 2.6, we have for all $t \in [0, T]$ and all initial datas

$$\| v_\Delta(\cdot, t) - v(\cdot, t) \|_{C^0} \leq \beta_2 \sqrt{\Delta x}.$$

By Proposition 2.8, we have for each $t \in [\Delta t, T]$ and all initial datas

$$(4.1) \quad \frac{u_\Delta(x_{m+2}, t) - u_\Delta(x_m, t)}{2\Delta x} \leq E_\Delta^{k(t)}.$$

Since $u_\Delta(\cdot, t)$ has the average 0, we have

$$\begin{aligned} \sum_{m; k(t)} \{u_\Delta(x_{m+2}, t) - u_\Delta(x_m, t)\} &= \sum_{m: +} \{u_\Delta(x_{m+2}, t) - u_\Delta(x_m, t)\} \\ &\quad + \sum_{m: -} \{u_\Delta(x_{m+2}, t) - u_\Delta(x_m, t)\} \\ &= 0, \end{aligned}$$

where $\sum_{m: +}$ ($\sum_{m: -}$) stands for the summation with respect to m for which $u_\Delta(x_{m+2}, t) - u_\Delta(x_m, t) \geq 0$ ($u_\Delta(x_{m+2}, t) - u_\Delta(x_m, t) < 0$). Hence it follows from (4.1) that the total variation of $u_\Delta(\cdot, t)$ on \mathbb{T} is bounded:

$$(4.2) \quad \sum_{m; k(t)} |u_\Delta(x_{m+2}, t) - u_\Delta(x_m, t)| = 2 \sum_{m: +} \{u_\Delta(x_{m+2}, t) - u_\Delta(x_m, t)\} \leq 2E_\Delta^{k(t)}.$$

For any $\varepsilon > 0$, there exists $\tilde{\Delta} = (\tilde{\Delta}x, \tilde{\Delta}t)$ such that

$$\| u_{\tilde{\Delta}}(\cdot, t) - u(\cdot, t) \|_{L^1(\mathbb{T})} \leq \varepsilon.$$

In particular, we take such a $\tilde{\Delta} = (\tilde{\Delta}x, \tilde{\Delta}t)$ that satisfies $\tilde{\Delta}t/\tilde{\Delta}x = \Delta t/\Delta x$, $\tilde{\Delta}x \leq (\beta_2^{-1}\varepsilon)^4$, $\Delta x/\tilde{\Delta}x = 3^p$ for some $p \in \mathbb{N}$. The last relation guarantees that the points of discontinuity of u_Δ are also these of $u_{\tilde{\Delta}}$. Then we have

$$(4.3) \quad \begin{aligned} \| u_\Delta(\cdot, t) - u(\cdot, t) \|_{L^1(\mathbb{T})} &\leq \| u_{\tilde{\Delta}}(\cdot, t) - u_\Delta(\cdot, t) \|_{L^1(\mathbb{T})} + \varepsilon, \\ \| v_{\tilde{\Delta}}(\cdot, t) - v_\Delta(\cdot, t) \|_{C^0} &\leq \beta_2 \sqrt{\Delta x} + \beta_2 \sqrt{\tilde{\Delta}x} \leq 2\beta_2 \sqrt{\Delta x}. \end{aligned}$$

Now we estimate $\| u_{\tilde{\Delta}}(\cdot, t) - u_\Delta(\cdot, t) \|_{L^1(\mathbb{T})}$. We introduce $w_\Delta := u_{\tilde{\Delta}}(\cdot, t) - u_\Delta(\cdot, t)$, $\tilde{w}_\Delta := v_{\tilde{\Delta}}(\cdot, t) - v_\Delta(\cdot, t)$ and $\tilde{k}(t) := 3^p k(t)$. Let $x_m \in \tilde{\Delta}x\mathbb{Z}$ and $x_{m_0} := 0$ for $k(t) = \text{even}$ or $x_{m_0} := \tilde{\Delta}x$ for $k(t) = \text{odd}$. We divide $\tilde{\Delta}x\mathbb{Z}$ according to the sign of w_Δ : Define I_1, I_2, \dots, I_{n+1} as

$$\begin{aligned} I_1 &:= \{x_{m_0}, x_{m_0+2}, \dots, x_{m_1}\} \text{ on which } w_\Delta(x) \geq 0 (< 0), \\ I_2 &:= \{x_{m_1+2}, x_{m_1+4}, \dots, x_{m_2}\} \text{ on which } w_\Delta(x) < 0 (\geq 0), \\ I_3 &:= \{x_{m_2+2}, x_{m_2+4}, \dots, x_{m_3}\} \text{ on which } w_\Delta(x) \geq 0 (< 0), \\ &\quad \dots, \\ I_n &:= \{x_{m_{n-1}+2}, x_{m_{n-1}+4}, \dots, x_{m_n}\} \text{ on which } w_\Delta(x) < 0 (\geq 0), \\ I_{n+1} &:= \{x_{m_n+2}, x_{m_n+4}, \dots, x_{m_0} + 1\} \text{ on which } w_\Delta(x) \geq 0 (< 0), \end{aligned}$$

where $n = \text{even}$ and $x_{m_n} \leq x_{m_0} + 1 - 2\tilde{\Delta}x$. We redefine I_1 as

$$I_1 := \{x_{m_0}, x_{m_0+2}, \dots, x_{m_1}\} \text{ with } x_{m_0} := x_{m_n+2} - 1.$$

Note that $w_\Delta(x) \geq 0$ (< 0) on I_1 . Set $|I_1| := x_{m_1} - x_{m_0} + 2\tilde{\Delta}x$ and $|I_j| := x_{m_j} - x_{m_{j-1}+2} + 2\tilde{\Delta}x$ for $j > 1$. We have $\sum_{j=1}^n |I_j| = 1$. For each I_j on which $w_\Delta(x) \geq 0$ (< 0), we have $y^j \in I_j$ for which $w_\Delta(x)$ takes the maximum (minimum) within I_j . Suppose that $w_\Delta(x) \geq 0$ on I_1 . In the other case, the argument is parallel. Note that

$$\|w_\Delta(x)\|_{L^1(\mathbb{T})} = \sum_{j=1}^{n/2} \left\{ \sum_{x \in I_{2j-1}} w_\Delta(x) \cdot 2\tilde{\Delta}x - \sum_{x \in I_{2j}} w_\Delta(x) \cdot 2\tilde{\Delta}x \right\}.$$

Introduce $J := \{j \mid 0 \leq j \leq n/2, \max\{|I_{2j-1}|, |I_{2j}|\} < \Delta x^{1/4}\}$ and $\tilde{J} := \{j \mid 0 \leq j \leq n/2, \max\{|I_{2j-1}|, |I_{2j}|\} \geq \Delta x^{1/4}\}$. We have $\#\tilde{J} \cdot \Delta x^{1/4} \leq 1$ and $\#\tilde{J} \leq \Delta x^{-1/4}$. Therefore, noting $w_\Delta = (\tilde{w}_\Delta)_x$, (4.2) and (4.3), we obtain

$$\begin{aligned} \|w_\Delta(x)\|_{L^1(\mathbb{T})} &= \sum_{j \in J} \left\{ \sum_{x \in I_{2j-1}} w(x) \cdot 2\tilde{\Delta}x - \sum_{x \in I_{2j}} w_\Delta(x) \cdot 2\tilde{\Delta}x \right\} \\ &\quad + \sum_{j \in \tilde{J}} \left\{ \sum_{x \in I_{2j-1}} w_\Delta(x) \cdot 2\tilde{\Delta}x - \sum_{x \in I_{2j}} w_\Delta(x) \cdot 2\tilde{\Delta}x \right\} \\ &\leq \sum_{j \in J} |w_\Delta(y^{2j-1}) - w_\Delta(y^{2j})| \Delta x^{\frac{1}{4}} \\ &\quad + \sum_{j \in \tilde{J}} \left[\{\tilde{w}_\Delta(x_{m_{2j-1}} + \tilde{\Delta}x) - \tilde{w}_\Delta(x_{m_{2j-2}+2} - \tilde{\Delta}x)\} \right. \\ &\quad \left. - \{\tilde{w}_\Delta(x_{m_{2j}} + \tilde{\Delta}x) - \tilde{w}_\Delta(x_{m_{2j-1}+2} - \tilde{\Delta}x)\} \right] \\ &\leq (2E_{\tilde{\Delta}}^{k(t)} + 2E_{\Delta}^{k(t)}) \Delta x^{\frac{1}{4}} + \#\tilde{J} \cdot 4 \cdot 2\beta_2 \sqrt{\Delta x} \\ &\leq 4E_{\Delta}^{k(t)} \Delta x^{\frac{1}{4}} + 8\beta_2 \Delta x^{\frac{1}{4}}, \end{aligned}$$

Since ε is arbitrary, we conclude that

$$\|u_\Delta(\cdot, t) - u(\cdot, t)\|_{L^1(\mathbb{T})} \leq (4E_{\Delta}^{k(t)} + 8\beta_2) \Delta x^{\frac{1}{4}}.$$

If v^0 satisfies the one-sided Lipschitz condition, $E_{\Delta}^{k(t)} \leq \max\{M, E^*\}$ for all t .

2. Note that $u = v_x$ and v is $C^{1,1}$ with $|v_\Delta(x, t) - v(x, t)| \leq \beta_2 \sqrt{\Delta x}$ on $\mathbb{T} \times [0, T]$. Fix arbitrary $t \in [0, T]$. We have for any $x, x' \in \mathbb{T}$

$$\left| \int_{x'}^x u_\Delta(y, t) - u(y, t) dy \right| \leq 2\beta_2 \sqrt{\Delta x}.$$

We already know that the step function $u_\Delta(\cdot, t)$ satisfies the one-sided Lipschitz condition. Therefore the linear interpolation of $u_m^{k(t)}$ with respect to the space variable, denoted by $\tilde{u}_\Delta(x)$, satisfies $\|\tilde{u}_\Delta - u_\Delta(\cdot, t)\|_{L^1(\mathbb{T})} \leq b_1 \Delta x$. Setting $w_\Delta := \tilde{u}_\Delta - u(\cdot, t)$, we have for all $x, x' \in \mathbb{T}$

$$(4.4) \quad \left| \int_{x'}^x w_\Delta(y) dy \right| \leq b_2 \sqrt{\Delta x}.$$

Since u is Lipschitz, w_Δ still satisfies the one-sided Lipschitz condition

$$\frac{w_\Delta(x_1) - w_\Delta(x_2)}{x_1 - x_2} \leq b_3.$$

Note that w_Δ does not necessarily satisfies the Lipschitz condition, because \tilde{u}_Δ does not necessarily satisfy the Lipschitz condition. Suppose that $|w_\Delta(\bar{x})| > b_4\Delta x^{\frac{1}{4}}$ with $(b_4)^2/(4b_2) > b_3$ for some \bar{x} . Let $I \ni \bar{x}$ be a connected interval on whose boundary we have $|w_\Delta(x)| = \frac{b_4}{2}\Delta x^{\frac{1}{4}}$. By (4.4), we see that

$$|I| \leq \frac{2b_2}{b_4}\Delta x^{\frac{1}{4}}.$$

If $w_\Delta(\bar{x}) > 0$ (< 0), we have with the left (right) boundary of I denoted by x

$$\frac{w_\Delta(\bar{x}) - w_\Delta(x)}{\bar{x} - x} \geq \frac{(b_4)^2}{4b_2} > b_3 \quad \left(\frac{w_\Delta(x) - w_\Delta(\bar{x})}{x - \bar{x}} \geq \frac{(b_4)^2}{4b_2} > b_3 \right),$$

which is a contradiction. Therefore we obtain

$$\|w_\Delta\|_{C^0} \leq b_4\Delta x^{\frac{1}{4}}.$$

Since $|u_\Delta(x, t) - u(x, t)| = |u_m^{k(t)} - u(x, t)| \leq |u_m^{k(t)} - u(x_m, t)| + b_5\Delta x = |w_\Delta(x_m)| + b_5\Delta x$, we conclude the theorem. \square

4.2 Error Estimate for KAM Tori

Let $\bar{u}^c = \bar{v}_x^c$ be a \mathbb{Z}^2 -periodic entropy solution of the C^1 -class. We remark the relation between such a \bar{u}^c and Hamiltonian dynamics: Consider the time-1 map $f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{T} \times \mathbb{R}$ of the Hamiltonian flow generated by the flux function $H(x, t, p)$ with the initial time equal to 0. Then $\{(x, c + \bar{u}^c(x, 0)) \mid x \in \mathbb{T}\} \cong \mathbb{T}$ is a smooth invariant torus of f . Due to the classical result of Poincaré, there exists a rotation number ω_1 . Let us regard the non-autonomous Hamiltonian dynamics generated by $H(x, t, p)$ as the autonomous one generated by $\mathcal{H}(q_1, q_2, p_1, p_2) := p_2 + H(q_1, q_2, p_1)$ in the extended phase space $\mathbb{T}^2 \times \mathbb{R}^2$. Define

$$\mathcal{I}(\bar{u}^c) := \{(q, g(q)) \mid q = (q_1, q_2) \in \mathbb{T}^2\} \cong \mathbb{T}^2,$$

where $g(q) := (c + \bar{u}^c(q_1, q_2), \bar{h}(c) - H(q_1, q_2, c + \bar{u}^c(q_1, q_2)))$. Then $\mathcal{I}(\bar{u}^c)$ is a smooth invariant torus of the Hamiltonian flow $\varphi_{\mathcal{H}}^s$ generated by \mathcal{H} . Let $C(s) := (\gamma^*(s), s)$ be the characteristic curves of \bar{u}^c , which satisfy $\gamma^{*'}(s) = H_p(\gamma^*(s), s, c + \bar{u}^c(\gamma^*(s), s))$ for $s \in \mathbb{R}$. The dynamics of the reduced characteristic curves $C^*(s) := C(s) \bmod 1 = (\gamma^*(s) \bmod 1, s \bmod 1)$ and that of the trajectories on $\mathcal{I}(\bar{u}^c)$ are identical, namely we have for all $s \in \mathbb{R}$

$$\varphi_{\mathcal{H}}^s(C^*(0), g(C^*(0))) = (C^*(s), g(C^*(s))).$$

Due to the classical result of Poincaré, $C(s)/s$ converges to $\omega = (\omega_1, 1) \in \mathbb{R}^2$ independent of $C(0)$ as $|s| \rightarrow \infty$. This ω is called a rotation vector of $\mathcal{I}(\bar{u}^c)$. If the rotation vector is irrational, each trajectory starting from a point of $\mathcal{I}(\bar{u}^c)$ is dense on $\mathcal{I}(\bar{u}^c)$. Therefore we obtain information on \bar{u}^c from only one characteristic curve, which is the key idea of the following argument. Approximation of \bar{u}^c leads to that of the invariant torus $\mathcal{I}(\bar{u}^c)$.

Now we consider a special case where $\mathcal{I}(\bar{u}^c)$ is a KAM torus: We say that c is associated with a KAM torus, if \bar{u}^c is C^1 and the dynamics of $C^*(s)$ is C^1 -conjugate to that of a linear flow on \mathbb{T}^2 with a Diophantine rotation vector, namely there exists a diffeomorphism $F : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ such that

$$C^*(s) = F(\omega s + \theta),$$

where $\theta \in \mathbb{R}$ depends on $C^*(0)$ and $\omega \in \mathbb{R}^2$ satisfies the ν, τ -Diophantine condition

$$|\omega_1 z_1 + \omega_2 z_2| \geq \nu \|z\|_1^{-\tau} \quad \text{for all } z \in \mathbb{Z}^2 \setminus \{0\}.$$

If c is associated with a KAM torus, \bar{u}^c is the unique \mathbb{Z}^2 -periodic entropy solution of (1.1) with the c . For the existence of a value c associated with a KAM torus, we refer to the KAM theory to the autonomous Hamiltonian systems with two-degree of freedom generated by the above $\mathcal{H}(q_1, q_2, p_1, p_2)$. We remark that the KAM theory under Rüssmann's non-degenerate condition (see e.g. [16]) works for such a degenerate \mathcal{H} in (p_1, p_2) with additional assumptions. The following error estimates hold:

Theorem 4.2. *Let $\Delta = (\Delta x, \Delta t)$ satisfy the conditions in Theorem 2.6, Proposition 2.8 and Theorem 3.1. Suppose that c is associated with a KAM torus. Let \bar{v}^c be a \mathbb{Z}^2 -periodic viscosity solution such that $\bar{v}_x^c = \bar{u}^c$. Then we have the space-time periodic difference solutions \bar{v}_Δ^c and \bar{u}_Δ^c that satisfy*

$$\sup_{(x,t) \in \mathbb{T}^2} |\bar{v}_\Delta^c(x, t) - \bar{v}^c(x, t)| \leq \beta_7 \Delta x^{\frac{1}{2(1+\tau)}}, \quad \sup_{(x,t) \in \mathbb{T}^2} |\bar{u}_\Delta^c(x, t) - \bar{u}^c(x, t)| \leq \beta_8 \Delta x^{\frac{1}{4(1+\tau)}},$$

where β_7 and β_8 are independent of Δ .

Proof. Let \bar{v}_Δ^c be a periodic difference solution. We omit c in \bar{v}^c , \bar{u}^c , etc. Fix an arbitrary $t \in \mathbb{T}$. By adding a constant to \bar{v}_Δ , if necessary, it holds that $\bar{v}_\Delta(x, t) - \bar{v}(x, t) \leq 0$ for all $x \in \mathbb{T}$ and $\bar{v}_\Delta(x^*, t) - \bar{v}(x^*, t) = 0$ for some $x^* \in \mathbb{T}$. Then we have n^* and l such that

$$0 = \bar{v}_\Delta(x^*, t) - \bar{v}(x^*, t) \leq \bar{v}_{n^*+1}^l - \bar{v}(x_{n^*+1}, t_l) + b_1 \Delta x,$$

where $|x^* - x_{n^*+1}| \leq 2\Delta x$ and $t \in [t_l, t_{l+1})$. For any $j \in \mathbb{N}$, we have a minimizing curve γ^* such that

$$\bar{v}(x_{n^*+1}, t_l) = \int_{-j+t_l}^{t_l} L^c(\gamma^*(s), s, \gamma^{*'}(s)) ds + \bar{v}(\gamma^*(-j+t_l), -j+t_l) + \bar{h}(c)j,$$

$$\bar{v}_{n^*+1}^l \leq \int_{-j+t_l}^{t_l} L^c(\gamma^*(s), s, \gamma^{*'}(s)) ds + \bar{v}_\Delta(\gamma^*(-j+t_l), -j+t_l) + \bar{h}_\Delta(c)j + b_2 \sqrt{\Delta x}j,$$

where we use an estimate similar to (3.5). Hence we obtain with 5. of Theorem 3.7

$$\begin{aligned} 0 &\leq \bar{v}_{n^*+1}^l - \bar{v}(x_{n^*+1}, t_l) + b_1 \Delta x \\ &\leq \bar{v}_\Delta(\gamma^*(-j+t_l), -j+t_l) - \bar{v}(\gamma^*(-j+t_l), -j+t_l) \\ &\quad + (\bar{h}_\Delta(c) - \bar{h}(c))j + b_2 \sqrt{\Delta x}j + b_1 \Delta x \\ &\leq \bar{v}_\Delta(\gamma^*(-j+t), t) - \bar{v}(\gamma^*(-j+t), t) + b_3 \sqrt{\Delta x}j. \end{aligned}$$

Since $\bar{v}_\Delta(x, t) - \bar{v}(x, t) \leq 0$ for all $x \in \mathbb{T}$, we obtain

$$|\bar{v}_\Delta(\gamma^*(-j+t), t) - \bar{v}(\gamma^*(-j+t), t)| \leq b_3 \sqrt{\Delta x}j.$$

Since $C^*(-s+t) := (\gamma^*(-s+t), -s+t) \bmod 1$ is a reduced characteristic curve, we have $C^*(-s+t) = F(\omega(-s+t) + \theta)$. It follows from [6] and [3] that for the set

$$\mathcal{N}_\varepsilon := \{\theta + \omega(-s+t) \bmod 1 \mid 0 \leq s \leq \frac{b_4}{\varepsilon^\tau}\}$$

is ε -dense on \mathbb{T}^2 , namely

$$\bigcup_{\zeta \in \mathcal{N}_\varepsilon} B_\varepsilon(\zeta) = \mathbb{T}^2,$$

where $B_\varepsilon(\zeta) = \{\tilde{\zeta} \in \mathbb{T}^2 \mid \|\tilde{\zeta} - \zeta\|_1 \leq \varepsilon\}$. Define $\mathcal{T} := F^{-1}(\mathbb{T} \times \{t\})$ and $X := (x, t)$ for $x \in \mathbb{T}$. For each X , we have $\zeta \in \mathcal{N}_\varepsilon \cap \mathcal{T}$ such that $\|\tilde{X} - \zeta\|_1 \leq \varepsilon$ with $\tilde{X} := F^{-1}(X)$ and $\zeta = \omega(-s^* + t) + \theta \pmod{1}$ with some $0 \leq s^* \leq \frac{b_4}{\varepsilon^\tau}$. Note that s^* must be an integer, because $F(\zeta) = C^*(-s^* + t) \in \mathbb{T} \times \{t\}$ and $-s^* + t \pmod{1} = t$. Hence, denoting $s^* = j$, we have

$$\|X - C^*(-j + t)\|_1 = \|F(\tilde{X}) - F(\zeta)\|_1 \leq \|DF\|_{op} \varepsilon.$$

Therefore we obtain for all $x \in \mathbb{T}$

$$\begin{aligned} |\bar{v}_\Delta(x, t) - \bar{v}(x, t)| &\leq |\bar{v}_\Delta(F(\tilde{X})) - \bar{v}_\Delta(F(\zeta))| + |\bar{v}_\Delta(F(\zeta)) - \bar{v}(F(\zeta))| \\ &\quad + |\bar{v}(F(\zeta)) - \bar{v}(F(\tilde{X}))| \\ &\leq b_5 \varepsilon + b_3 \sqrt{\Delta x} j + b_5 \varepsilon \\ &\leq b_6 \left(\frac{\sqrt{\Delta x}}{\varepsilon^\tau} + \varepsilon \right). \end{aligned}$$

Taking $\varepsilon = \Delta x^{\frac{1}{2(1+\tau)}}$, we have for all $x \in \mathbb{T}$

$$|\bar{v}_\Delta(x, t) - \bar{v}(x, t)| \leq 2b_6 \Delta x^{\frac{1}{2(1+\tau)}}.$$

Note that b_6 is independent of the choice of t . We have for $\bar{u}_\Delta = (\bar{v}_\Delta)_x$, $\bar{u} = \bar{v}_x$ and all $x, x' \in \mathbb{T}$

$$\left| \int_{x'}^x \bar{u}_\Delta(y, t) - \bar{u}(y, t) dy \right| \leq 4b_6 \Delta x^{\frac{1}{2(1+\tau)}}.$$

Since \bar{u}_Δ satisfies the one-sided Lipschitz condition, the linear interpolation of \bar{u}_m^l with respect to the space variable, denoted by $\tilde{u}_\Delta(x)$, satisfies $\|\tilde{u}_\Delta - \bar{u}_\Delta(\cdot, t)\|_{L^1(\mathbb{T})} \leq b_7 \Delta x$. Setting $w_\Delta := \tilde{u}_\Delta - \bar{u}(\cdot, t)$, we have for all $x, x' \in \mathbb{T}$

$$(4.5) \quad \left| \int_{x'}^x w_\Delta(y) dy \right| \leq b_8 \Delta x^{\frac{1}{2(1+\tau)}}.$$

Since \bar{u} is C^1 , w_Δ still satisfies the one-sided Lipschitz condition

$$\frac{w_\Delta(x_1) - w_\Delta(x_2)}{x_1 - x_2} \leq b_9.$$

Suppose that $|w_\Delta(\bar{x})| > b_{10} \Delta x^{\frac{1}{4(1+\tau)}}$ with $(b_{10})^2 / (4b_8) > b_9$ for some \bar{x} . Let $I \ni \bar{x}$ be a connected interval on whose boundary $|w_\Delta(x)| = \frac{b_{10}}{2} \Delta x^{\frac{1}{4(1+\tau)}}$. By (4.5), we see that

$$|I| \leq \frac{2b_8}{b_{10}} \Delta x^{\frac{1}{4(1+\tau)}}.$$

If $w_\Delta(\bar{x}) > 0$ (< 0), we have with the left (right) boundary of I denoted by x

$$\frac{w_\Delta(\bar{x}) - w_\Delta(x)}{\bar{x} - x} \geq \frac{(b_{10})^2}{4b_8} > b_9 \quad \left(\frac{w_\Delta(x) - w_\Delta(\bar{x})}{x - \bar{x}} \geq \frac{(b_{10})^2}{4b_8} > b_9 \right),$$

which is a contradiction. Therefore we obtain

$$\|w_\Delta\|_{C^0} \leq b_{10} \Delta x^{\frac{1}{4(1+\tau)}}.$$

Since $|\bar{u}_\Delta(x, t) - \bar{u}(x, t)| = |\bar{u}_m^l - \bar{u}(x, t)| \leq |\bar{u}_m^l - \bar{u}(x_m, t)| + b_{11} \Delta x = |w_\Delta(x_m)| + b_{11} \Delta x$, we conclude the theorem. \square

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